An introduction to globally symmetric spaces

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Introduction

The present text provides lecture notes for a course on symmetric spaces given in the framework of the "Semaine spéciale M2 : Géométrie et théorie des groupes" held at the Institut de Recherche Mathématique Avancée in Strasbourg from April 28 to

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May 3, 2008. It intends to give an accessible introduction to the theory of Riemannian symmetric spaces with an emphasis on those of non-compact type. Since the excellent textbook [H] by S. Helgason on the subject contains complete proofs of all relevant results way beyond the scope of this introduction, we content ourselves here with merely stating certain results, giving precise references for the more ambitious reader. We want to give the reader a guideline through a part of the landscape, trying to motivate the steps we take and illustrating the basic results by means of a detailed treatment of particular important examples. For a deeper understanding, the reader is strongly encouraged to study the books by Helgason [H], Eberlein [E] and also Borel [B1] and Wolf [Wo].

The plan of the text is as follows: Section 1 gives an overview on the geometry and algebraic coding of arbitrary globally symmetric spaces. In Section 2 we investigate more precisely the case of symmetric spaces of non-compact type which, in particular, are manifolds of non-positive sectional curvature. Their theory is intimately related to the theory of semi-simple Lie groups, so we describe the important Iwasawa and Cartan decompositions of such spaces. Finally in Section 3 we study the geometry at infinity of globally symmetric spaces of non-compact type: like any Hadamard manifold these spaces can be compactified by adding a sphere at infinity. Due to the rich structure of symmetric spaces, this geometric boundary can be described more precisely: we give a parametrization of boundary points in terms of the Cartan decomposition, relate it to the Furstenberg boundary and show how the Bruhat decomposition helps to describe pairs of boundary points which can be joined by a geodesic. In the last section we study Busemann functions and see how they can be used to obtain invariant Finsler metrics on the differentiable manifold underlying the symmetric space.

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1 Generalities on symmetric spaces

In this section we begin with a definition of Riemannian symmetric spaces and deduce many important properties from it. We will see that such manifolds have a huge group of isometries which acts transitively. Moreover, any simply connected symmetric space S is diffeomorphic to a homogeneous space G/K, where G is a connected Lie group with an involutive automorphism whose fixed point set is essentially the compact subgroup $K \subset G$.

This algebraic coding allows to describe the geometry in Lie algebraic terms: we will see that geodesics are projections to G/K of certain one-parameter subgroups of G, the curvature tensor is described by Lie brackets, and totally geodesic submanifolds correspond to Lie triple systems. In particular, the Levi-Civita connection remains the same when endowing the differentiable manifold S with a different Riemannian structure with respect to which S is also a symmetric space.

Towards the end of this introductory section we will see that after dividing out a direct factor isomorphic to a Euclidean motion group, the isometry group G becomes semi-simple; in this way the problem is reduced to the study of certain involutive automorphisms of semi-simple Lie algebras.

1.1 Geometric definition

Let S be a connected Riemannian manifold and $x \in S$. The *geodesic symmetry* s_x at x is the local diffeomorphism defined by $s_x(y) := \exp_x\big(-\mathrm{id}_{T_xS}(\exp_x^{-1}(y))\big)$.

Definition 1.1. S is called *locally symmetric*, if s_x is a local isometry for all $x \in S$. If s_x is a global isometry for all $x \in S$, then S is called (*globally*) *symmetric*.

Examples. $S = \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ are globally symmetric, and any quotient $\Gamma\backslash S$, where $\Gamma \subset \mathrm{Is}(S)$ is a discrete, torsion free group of isometries of S, is locally symmetric.

Theorem 1.2 ([H], Theorem IV.5.6). *A simply connected locally symmetric space is globally symmetric.*

Notice that this theorem in particular implies that the Riemannian universal cover of a locally symmetric space is globally symmetric. Conversely, every locally symmetric space is a quotient of a globally symmetric space by a discrete, torsion free group of isometries isomorphic to the fundamental group. In these notes we will only be concerned with globally symmetric spaces. Let d denote the distance function on S induced from the Riemannian metric.

Proposition 1.3. *If S is globally symmetric, then S is complete and homogeneous.*

Proof. For completeness we show that all geodesics are defined on $\mathbb{R}$. The claim then follows from the theorem of Hopf–Rinow.

Let c be a geodesic in S and suppose there exists $b \in \mathbb{R}$ such that c is defined on (a,b) for some $a < b$, but not at b. Take $\varepsilon = \frac{b-a}{4}$ and consider the geodesic symmetry s_x at $x := c(b-\varepsilon)$. Then $c(b) = s_x(c(b-2\varepsilon))$ exists, hence c is defined at parameter b which contradicts our assumption.

By the Theorem of Hopf–Rinow and completeness we know that for any pair of points $x, y \in S$ and $t := d(x,y)$ there exists a geodesic $c\colon \mathbb{R} \to S$ such that $c(0) = x$ and $c(t) = y$. Then $y = s_{c(t/2)} \circ s_x(x)$, i.e. S is homogeneous. $\square$

Remark 1.4. Notice that the isometry $s_{c(t/2)} \circ s_x$ in the above proof belongs to the connected component $\mathrm{Is}^o(S)$ of the identity in $\mathrm{Is}(S)$. Hence we have shown that the (possibly smaller) group $\mathrm{Is}^o(S)$ acts transitively on S.

For $x, y \in S$ we denote by $c_{x,y}$ the unique unit speed geodesic emanating from x which contains y. With this notation we have

$$s_x(y) = s_x\big(c_{x,y}(d(x,y))\big) = c_{x,y}(-d(x,y)). \tag{1.1}$$

1.2 The group of isometries

We first state an important rigidity property of isometries of a Riemannian manifold which we will need in the sequel. For a diffeomorphism Φ of a Riemannian manifold M we denote by $D\Phi\colon TM \to TM$ its differential.

Lemma 1.5 (Rigidity of isometries; [dC], Lemma 4.2). *Let Φ and Ψ be two local isometries of a connected Riemannian manifold S. Assume that at some point x we have $\Phi(x) = \Psi(x)$ and $D_x\Phi = D_x\Psi$. Then $\Phi = \Psi$.*

Moreover, the group of isometries of a Riemannian manifold satisfies the following properties:

Theorem 1.6 ([H], Theorem IV.2.5). *Endowed with the compact-open topology the isometry group $\mathrm{Is}(S)$ of a Riemannian manifold S is a locally compact topological transformation group of S. Moreover, for all $x \in S$ the isotropy subgroup $\mathrm{Is}(S)_x := \{g \in \mathrm{Is}(S) : g(x) = x\}$ at x is compact.*

In the sequel we assume that S is globally symmetric, and denote by $G := \mathrm{Is}^o(S)$ the identity component of $\mathrm{Is}(S)$. We fix $o \in S$ and let $K := \{g \in \mathrm{Is}^o(S) : g(o) = o\}$ be the compact isotropy subgroup of G at o. Then by Remark 1.4 we have $S = G(o) := \{g(o) : g \in G\}$.

Theorem 1.7 ([H], Lemma IV.3.2 and Theorem IV.3.3 (i), (ii)). *The topological group G has an analytic structure compatible with the compact-open topology in which it is a connected Lie transformation group of S. Moreover, G/K is analytically diffeomorphic to S, and K contains no non-trivial normal subgroup of G.*

Notice that by Theorem II.2.6 of [H] a topological group has at most one analytic structure compatible with its topology with respect to which it is a Lie group.

In the remainder of this section we will have a look at the geodesic symmetry in S.

Lemma 1.8. *If $x \in S$ and $k \in \mathrm{Is}(S)_x$ then the geodesic symmetry at x satisfies*

$$s_x \circ k = k \circ s_x.$$

Proof. Let $z \in S$ arbitrary. If $t := d(x,z) = d(k(x),k(z)) = d(x,k(z))$, then by (1.1) and the fact that $g(c_{x,z}) = c_{g(x),g(z)}$ for any isometry $g \in \mathrm{Is}(S)$ we get $s_x(k(z)) = c_{x,k(z)}(-t) = k(c_{x,z}(-t)) = k(s_x(z))$. □

Lemma 1.9. *Let $x \in S$ and $g \in \mathrm{Is}(S)$ be such that $x = g(o)$. Then $s_x = g \circ s_o \circ g^{-1}$.*

Proof. Let $y \in S$ arbitrary, $t := d(x, y)$ and $z := g^{-1}(y)$. Then

$$s_x(y) = c_{x,y}(-t) = c_{g(o),g(z)}(-t) = g(c_{o,z}(-t)) = g(s_o(c_{o,z}(t))),$$

hence the claim follows from $c_{o,z}(t) = z = g^{-1}(y)$. □

1.3 Algebraic point of view

We have seen that globally symmetric spaces are diffeomorphic to G/K, where G is a connected Lie group and $K \subset G$ the isotropy subgroup at some point. One natural question concerns the reverse statement: which homogeneous spaces are symmetric spaces?

Before we address this question we need some more facts relating a globally symmetric space S to the connected Lie group $G := \mathrm{Is}^o(S)$ and its Lie algebra $\mathfrak{g}$. Denote $e\colon \mathfrak{g} \to G$ the exponential mapping of $\mathfrak{g}$ into G, and $e \in G$ the identity element in G. We fix a base point $o \in S$, let $K \subset G$ be the isotropy subgroup of G at o, and consider the geodesic symmetry s_o at o. The automorphism $\sigma \in \mathrm{Aut}(G)$ of G defined by $\sigma(g) := s_o \circ g \circ s_o^{-1}$ is an involution, i.e. σ^2 is the identity $\mathrm{id}_G \in \mathrm{Aut}(G)$. We set

$$G^\sigma := \{g \in G : \sigma(g) = g\},$$

and we denote by $(G^\sigma)^o$ the identity component of G^σ.

Notation. For simplification, we will in the sequel omit the "$\circ$" when referring to composition of group elements in G. Moreover, the action of G on S will be denoted by a dot "$\cdot$".

Proposition 1.10. $(G^\sigma)^o \subseteq K \subseteq G^\sigma$.

Proof. Let $k \in K$. Then $s_o k\, s_o^{-1} \cdot o = o = k \cdot o$ and

$$D_o(s_o k\, s_o^{-1}) = -\mathrm{id}_{T_oS} \circ D_o k \circ (-\mathrm{id}_{T_oS}) = D_o k,$$

hence by rigidity of isometries $s_o k\, s_o^{-1} = k$ and therefore $K \subset G^\sigma$.

Next let $g \in (G^\sigma)^o$. Then there exists a path $p\colon [0, 1] \to (G^\sigma)^o$ such that $p(0) = e$ and $p(1) = g$. Now $o = s_o \cdot o$ gives

$$s_o p(t) \cdot o = s_o p(t) s_o^{-1} \cdot o = \sigma(p(t)) \cdot o = p(t) \cdot o$$

for all $t \in [0, 1]$, i.e. $p(t) \cdot o$ is a fixed point of s_o for all $t \in [0, 1]$. But o is an isolated fixed point of s_o, hence necessarily $p(t) \cdot o = o$ for all $t \in [0, 1]$. In particular we have $g \cdot o = p(1) \cdot o = o$ which implies $g \in K$. □

In order to give a condition under which a homogeneous space is symmetric, we recall some facts from the theory of Lie groups and Lie algebras.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then for $h \in G$ the conjugation map $I(h)\colon G \to G$, $g \mapsto hgh^{-1}$ is an isomorphism of Lie groups. We denote by $\mathrm{Ad}(h) := D_e(I(h))\colon \mathfrak{g} \to \mathfrak{g}$ its differential at the identity $e \in G$. $\mathrm{Ad}(h)$ is a Lie algebra automorphism, hence in particular

$$[\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y] = \mathrm{Ad}(h)[X, Y] \quad \text{for all } X, Y \in \mathfrak{g}.$$

Moreover, we have the following useful formula:

$$e^{\mathrm{Ad}(h)X} = he^X h^{-1} \quad \text{for any } h \in G,\ X \in \mathfrak{g}. \tag{1.2}$$

The map $\mathrm{Ad}_G\colon G \to \mathrm{GL}(\mathfrak{g})$, $h \mapsto \mathrm{Ad}(h)$ is an analytic group morphism which is called the *adjoint representation of* G.

Definition 1.11. (G, K) is called a *Riemannian symmetric pair* if G is a connected Lie group, $K \subset G$ a closed subgroup such that $\mathrm{Ad}_G(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$ and if there exists an analytic involutive automorphism σ of G such that

$$(G^\sigma)^o \subseteq K \subseteq G^\sigma.$$

Notice that if S is a globally symmetric space, $G = \mathrm{Is}^o(S)$ and $K \subset G$ the isotropy subgroup of an arbitrary point $x \in S$, then (G, K) is a Riemannian symmetric pair with respect to the analytic involutive automorphism of G induced by the geodesic symmetry at x. In this case we call (G, K) the *Riemannian symmetric pair associated to* (S, x). The following theorem in particular answers the question raised in the introduction.

Proposition 1.12 ([H], Proposition IV.3.4). *If (G, K) is a Riemannian symmetric pair and σ any analytic involutive automorphism of G such that $(G^\sigma)^o \subseteq K \subseteq G^\sigma$, then G/K is a globally symmetric space with respect to any G-invariant Riemannian metric. If $\pi\colon G \to G/K$ denotes the natural projection and s_o the geodesic symmetry at $o = \pi(K) = eK \in G/K$, then*

$$s_o \circ \pi = \pi \circ \sigma.$$

In particular, s_o is independent of the choice of the G-invariant Riemannian metric.

The following proposition shows that under very general conditions the automorphism σ is completely determined by its set of fixed points G^σ.

Proposition 1.13 ([H], Proposition IV.3.). *Let (G, K) be a Riemannian symmetric pair, $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{z}$ the Lie algebra of the center of G. If $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, then there exists exactly one analytic involutive automorphism σ of G such that $(G^\sigma)^o \subseteq K \subseteq G^\sigma$.*

We remark that for semi-simple Lie groups G we have $\mathfrak{z} = \{0\}$, hence clearly $\mathfrak{k} \cap \mathfrak{z} = \{0\}$. Moreover, if (G, K) is the Riemannian symmetric pair associated to a globally symmetric space S with base point $o \in S$, then K contains no non-trivial normal subgroup of G by Theorem 1.7; hence in this case the analytic involutive automorphism σ of G induced by the geodesic symmetry at o is the only one satisfying $(G^\sigma)^o \subseteq K \subseteq G^\sigma$.

We next look at the *Cartan involution* $\Theta\colon \mathfrak{g} \to \mathfrak{g}$ defined as the differential $\Theta := D_e\sigma$ of σ at the identity $e \in G$. Since $\Theta^2 = \mathrm{Id}_{\mathfrak{g}}$ one can look at the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ with respect to the eigenvalues $+1$ and -1 respectively. This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to Θ.

Moreover, Θ is a Lie algebra automorphism and we have the *Cartan relations*

Lemma 1.14. $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Proof. We prove $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, the other inclusions are similar. Let $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ arbitrary. Then

$$\Theta[X, Y] = [\Theta X, \Theta Y] = [X, -Y] = -[X, Y],$$

i.e. $[X, Y]$ belongs to the -1-eigenspace of Θ. $\square$

1.4 Geodesics and curvature

Now let S be a globally symmetric space with base point $o \in S$ and (G, K) the associated Riemannian symmetric pair. By our remark following Proposition 1.13 there exists exactly one analytic involutive automorphism σ of G with $(G^\sigma)^o \subseteq K \subseteq G^\sigma$, so the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is uniquely determined. Let $\tau\colon G \to S$, $g \mapsto g \cdot o$, denote the natural map, $e\colon \mathfrak{g} \to G$ the Lie group exponential mapping, $D_e\tau\colon \mathfrak{g} \to T_oS$ the differential of τ at the identity $e \in G$, and $\exp_o\colon T_oS \to S$ the Riemannian exponential mapping. The importance of the Cartan decomposition of $\mathfrak{g}$ is reflected in the following

Theorem 1.15 ([H], Theorem IV.3.3 (iii)). *$D_e\tau|_{\mathfrak{p}}\colon \mathfrak{p} \to T_oS$ is an isomorphism (of vector spaces with Lie bracket), and* $\ker(D_e\tau) = \mathfrak{k}$. *Moreover, we have*

$$\tau(e^X) = \exp_o\big(D_e\tau(X)\big) \quad \textit{for any } X \in \mathfrak{p}. \tag{1.3}$$

This shows in particular that the Riemannian exponential map $\exp\colon TS \to S$ of a globally symmetric space does not depend on its Riemannian metric; for any G-invariant Riemannian metric on $S \cong G/K$ the exponential map is the same! Moreover, this immediately shows how geodesics in S look like:

Corollary 1.16. *The geodesic $c \subset S$ emanating from o with tangent vector $D_e\tau(X) \in T_oS$, $X \in \mathfrak{p}$, is given by*

$$c(t) = e^{tX} \cdot o, \quad t \in \mathbb{R}.$$

Notice that if $c \subset S$ is an arbitrary geodesic, then by transitivity of the action of G there exists $g \in G$ such that $c(0) = g \cdot o$. So $g^{-1} \cdot c$ is a geodesic emanating from $o \in S$ and therefore of the form $e^{tX} \cdot o$ for some $X \in \mathfrak{p}$. This shows that for every geodesic $c \subset S$ there exist $g \in G$ and $X \in \mathfrak{p}$ such that

$$c(t) = ge^{tX} \cdot o, \quad t \in \mathbb{R}.$$

The following theorem describes the curvature tensor and totally geodesic submanifolds of S. Notice that for the curvature tensor we use the definition from [H]; in the book [dC] by do Carmo the curvature tensor is defined with the opposite sign.

Theorem 1.17 ([H], Theorem IV.4.2, Theorem IV.7.2). (1) *The curvature tensor R_o evaluated in T_oS is given by*

$$R_o(D_e\tau(X), D_e\tau(Y))D_e\tau(Z) = D_e\tau\big(-[[X,Y],Z]\big), \quad X,Y,Z \in \mathfrak{p}.$$

(2) *Totally geodesic submanifolds through o are of the form $e^{\mathfrak{q}} \cdot o$, where $\mathfrak{q} \subseteq \mathfrak{p}$ is a Lie triple system, i.e.* $[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subseteq \mathfrak{q}$.

In particular – as we have already seen for the Riemannian exponential mapping $\exp\colon TS \to S$ – the curvature tensor and the totally geodesic submanifolds of S do not depend on the given Riemannian metric on S. These facts also follow from the following

Theorem 1.18 ([H], Corollary IV.4.3). *The Levi-Civita connection on G/K is the same for all G-invariant Riemannian structures on G/K.*

1.5 Examples

(1) $\mathbf{SL}(n,\mathbb{R})/\mathbf{SO}(n)$.

Consider the connected Lie group $G = \mathrm{SL}(n,\mathbb{R})$ which is the group of all $(n\times n)$-matrices with determinant 1 and entries in $\mathbb{R}$. On G we consider the involutive automorphism $\sigma\colon G \to G$, $g \mapsto (g^t)^{-1}$. Then

$$G^\sigma = \{g \in G : (g^t)^{-1} = g\} = \{g \in G : g^t g = e\} = \mathrm{SO}(n) =: K.$$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ consists of all $(n \times n)$-matrices with trace 0 and entries in $\mathbb{R}$, and the Cartan involution $\Theta\colon \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R})$ is given by $\Theta(X) := -X^t$ for $X \in \mathfrak{sl}(n,\mathbb{R})$. So the Cartan decomposition of an element $X \in \mathfrak{sl}(n,\mathbb{R})$ is the well-known unique decomposition

$$X = \tfrac{1}{2}(X - X^t) + \tfrac{1}{2}(X + X^t)$$

of a matrix into its anti-symmetric and symmetric part. If $\mathfrak{so}(n)$ denotes the Lie algebra of $K = \mathrm{SO}(n)$, and $\mathrm{sym}_0(n)$ the set of symmetric $(n \times n)$-matrices of trace zero with entries in $\mathbb{R}$, we therefore have

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{sym}_0(n).$$

Denote $o = eK \in G/K$ the base point and consider the positive definite symmetric bilinear form

$$\langle X, Y \rangle := \mathrm{Tr}(X \cdot Y), \quad X, Y \in T_o(G/K) \cong \mathfrak{p} \subseteq \mathfrak{g}. \tag{1.4}$$

As we will see more precisely in Section 1.7, this scalar product on $T_o(G/K)$ can be naturally extended by G-left-translations to a G-invariant Riemannian metric on G/K.

The set $\mathrm{Pos}_1(n)$ of positive definite symmetric $(n \times n)$-matrices with determinant 1 and entries in $\mathbb{R}$ can be identified with G/K as follows: it is a standard fact from elementary linear algebra that any matrix $p \in \mathrm{Pos}_1(n)$ can be written as a matrix product $p = b^t b$ for some $b \in \mathrm{SL}(n, \mathbb{R})$. With the action of $g \in \mathrm{SL}(n, \mathbb{R})$ on $\mathrm{Pos}_1(n)$ given by $g \cdot p := g^t p g$, $p \in \mathrm{Pos}_1(n)$, G acts transitively on $\mathrm{Pos}_1(n)$. If we choose the $(n \times n)$-identity matrix I_n as a base point o in $\mathrm{Pos}_1(n)$, then $\mathrm{SO}(n) = \{g \in G : g \cdot I_n = I_n\}$.

If $n = 2$, we can identify G/K endowed with the G-invariant Riemannian metric induced by $\langle X, Y \rangle := 2 \cdot \mathrm{Tr}(X \cdot Y)$, $X, Y \in T_o(G/K)$, and the real hyperbolic plane $\mathbb{H}^2 := \{x + iy : x \in \mathbb{R} \;\; y > 0\}$ endowed with the metric $ds^2 = (dx^2 + dy^2)/y^2$. Indeed, $\mathrm{SL}(2, \mathbb{R})$ acts transitively by isometries via linear fractional transformations on $\mathbb{H}^2$, and $\mathrm{SO}(2)$ is the isotropy subgroup of the point $i \in \mathbb{H}^2$. So $\mathrm{Pos}_1(2)$ with a metric rescaled by the factor 2 can be identified with the hyperbolic plane $(\mathbb{H}^2, ds^2)$.

Exercise. Show that we need to have the factor 2 in equation (1.4) in order that $(\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2), \langle \cdot, \cdot \rangle)$ is isometric to $(\mathbb{H}^2, ds^2)$.

(2) G/K, $G \subset \mathrm{SL}(n, \mathbb{R})$ closed subgroup with $G^t = G$.

As involutive automorphism we take again $\sigma \colon G \to G$, $g \mapsto (g^t)^{-1}$, so $K = G \cap \mathrm{SO}(n)$. If $o = eK \in G/K$ denotes the base point, then the positive definite bilinear form given by (1.4) on $T_o(G/K)$ extends to a G-invariant Riemannian metric on G/K.

(a) The group $G = \mathrm{SO}(p, q)$ of linear transformations leaving invariant the bilinear form

$$Q(x, y) = -x_1 y_1 - \cdots - x_p y_p + x_{p+1} y_{p+1} + \cdots + x_{p+q} y_{p+q},$$

$x, y \in \mathbb{R}^{p+q}$, on $\mathbb{R}^{p+q}$ is invariant under transposition. Therefore if $K = G \cap \mathrm{SO}(p+q) = \mathrm{SO}(p) \times \mathrm{SO}(q)$ we get a symmetric space G/K.

Let $M(p, q)$ denote the set of $(p \times q)$-matrices with entries in $\mathbb{R}$. The Cartan decomposition of the Lie algebra of $\mathrm{SO}(p, q)$ is given by $\mathfrak{so}(p, q) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{so}(p) \times \mathfrak{so}(q) \subset \mathfrak{so}(p+q)$ is the Lie algebra of K and

$$\mathfrak{p} = \left\{ \left(\begin{smallmatrix} 0 & B \\ B^t & 0 \end{smallmatrix} \right) : B \in M(p, q) \right\} \subset \mathrm{sym}_0(p+q).$$

In the particular case $p = 1$, this symmetric space with an appropriately rescaled metric is isometric to the hyperbolic space of dimension q.

(b) The group $G = \mathrm{Sp}(2q, \mathbb{R})$ of linear transformations leaving invariant the standard symplectic form

$$\omega(x, y) = x_1 y_{q+1} + x_2 y_{q+2} + \cdots + x_q y_{2q} - x_{q+1} y_1 - \cdots - x_{2q} y_q,$$

$x, y \in \mathbb{R}^{2q}$, on $\mathbb{R}^{2q}$ is invariant under transposition. If $K = \mathrm{Sp}(2q, \mathbb{R}) \cap \mathrm{SO}(2q)$, then G/K is a symmetric space.

The Cartan decomposition of the Lie algebra of $\mathrm{Sp}(2q, \mathbb{R})$ is given by $\mathfrak{sp}(2q, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \left(\begin{smallmatrix} A & B \\ -B^t & A \end{smallmatrix}\right) : A, B \in M(q,q),\ A^t = -A \right\} \subset \mathfrak{so}(2q)$$

is the Lie algebra of K and

$$\mathfrak{p} = \left\{ \left(\begin{smallmatrix} A & B \\ B^t & -A \end{smallmatrix}\right) : A, B \in M(q,q),\ A^t = A \right\} \subset \mathrm{sym}_0(2q).$$

Recall that a complex structure on a real vector space V is an endomorphism J of V with the property $J^2 = -\mathrm{id}_V$. Moreover, if $g \in \mathrm{GL}(V)$, then $g \circ J \circ g^{-1}$ is also a complex structure.

Consider the set S_{2q} of complex structures J on the symplectic vector space $(\mathbb{R}^{2q}, \omega)$ such that the symmetric bilinear form defined by

$$q_J(x, y) := \omega(x, Jy), \quad x, y \in \mathbb{R}^{2q}, \tag{1.5}$$

is positive definite. A complex structure with this property is called *ω-compatible*. $G = \mathrm{Sp}(2q, \mathbb{R})$ acts naturally on S_{2q} by conjugation, i.e. $g \cdot J := gJg^{-1}$ for $g \in G$, $J \in S_{2q}$. Indeed, if $g \in \mathrm{Sp}(2q, \mathbb{R})$ then

$$q_{g \cdot J}(x, y) = \omega(x, gJg^{-1}y) = \omega(g^{-1}x, Jg^{-1}y) = q_J(g^{-1}x, g^{-1}y),$$

so $q_{g \cdot J}$ is positive definite if q_J is. Moreover, this action is transitive. We choose as a base point $o \in S_{2q}$ the ω-compatible complex structure given by the matrix

$$J_0 := \begin{pmatrix} 0 & -I_q \\ I_q & 0 \end{pmatrix}; \tag{1.6}$$

its associated symmetric bilinear form q_{J_0} is the standard scalar product in $\mathbb{R}^{2q}$. Then the isotropy subgroup of G at o is precisely the group $K = \mathrm{Sp}(2q, \mathbb{R}) \cap \mathrm{SO}(2q)$, so $S_{2q} = \mathrm{Sp}(2q, \mathbb{R}) \cdot o$ can be identified with G/K.

Notice that in the particular case $q = 1$ we have $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, so the subspace S_2 of $\mathbb{R}^2$ with the appropriately rescaled metric can be identified with the hyperbolic plane $(\mathbb{H}^2, ds^2)$.

(c) The group $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ acts by isometries on $\mathbb{H}^2 \times \mathbb{H}^2$ endowed with the product metric, and $K = \mathrm{SO}(2) \times \mathrm{SO}(2)$ fixes the point $o := (i, i) \in \mathbb{H}^2 \times \mathbb{H}^2$. So in this case the symmetric space G/K endowed with the G-invariant metric induced by $\langle X, Y\rangle := 2 \cdot \mathrm{Tr}(X \cdot Y)$, $X, Y \in T_o(G/K)$, is isometric to a product of hyperbolic planes $\mathbb{H}^2 \times \mathbb{H}^2$.

(3) $\mathbf{SO}(p+q)/(\mathbf{SO}(p) \times \mathbf{SO}(q))$.

As before we denote by I_q the $(q \times q)$-identity matrix and let $s \in \mathrm{SL}(p+q, \mathbb{R})$ be the matrix $s = \left(\begin{smallmatrix} -I_p & 0 \\ 0 & I_q \end{smallmatrix}\right)$. For $G = \mathrm{SO}(p+q)$ we consider the involutive automorphism $\sigma\colon G \to G$, $g \mapsto sgs$. Then $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ is a compact subgroup fixed by σ.

The Cartan involution $\Theta\colon \mathfrak{so}(p+q) \to \mathfrak{so}(p+q)$ is given as follows: if $A \in M(p,p)$, $A^t = -A$, $B \in M(p,q)$, $D \in M(q,q)$, $D^t = -D$, then for

$$X = \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} \in \mathfrak{so}(p+q)$$

we have

$$\Theta(X) = \begin{pmatrix} A & -B \\ B^t & D \end{pmatrix}.$$

So $\mathfrak{so}(p+q) = (\mathfrak{so}(p) \times \mathfrak{so}(q)) \oplus \mathfrak{p}$, where

$$\mathfrak{p} := \left\{ \left(\begin{smallmatrix} 0 & B \\ -B^t & 0 \end{smallmatrix}\right) : B \in M(p,q) \right\}.$$

In this case, the symmetric bilinear form given by

$$\langle X, Y\rangle := -\mathrm{Tr}(X \cdot Y), \quad X, Y \in T_o(G/K) \cong \mathfrak{p} \subseteq \mathfrak{g}, \tag{1.7}$$

is positive definite, and hence can be extended to a G-invariant Riemannian metric on G/K.

Here the symmetric space G/K is the Grassmannian manifold of p-dimensional oriented subspaces of $\mathbb{R}^{p+q}$. In the particular case $p = 1$ this is the q-dimensional sphere, and the standard metric induced from the embedding into $\mathbb{R}^{q+1}$ is a scalar multiple of the above metric.

(4) Compact Lie groups as symmetric spaces.

Let G be a compact connected Lie group. Then the mapping

$$\sigma\colon G \times G \to G \times G, \quad (g_1, g_2) \mapsto (g_2, g_1),$$

is an involutive automorphism of the product group $G \times G$. The fixed point set of σ is the diagonal $\Delta := \{(g,g) : g \in G\}$ in $G \times G$ which is isomorphic to G and hence compact. The pair $(G \times G, \Delta)$ is a Riemannian symmetric pair and the coset space $(G \times G)/\Delta$ is diffeomorphic to the original group G via the mapping $(G \times G)/\Delta \to G$, $(g_1, g_2)\Delta \mapsto g_1 g_2^{-1}$.

A Riemannian metric on $(G \times G)/\Delta$ is $(G \times G)$-invariant if and only if the corresponding Riemannian metric on the group G is bi-invariant. Hence by Proposition 1.12, G is a globally symmetric space with respect to any bi-invariant Riemannian metric on G. The natural mapping of $G \times G$ onto $G \cong (G \times G)/\Delta$

corresponds to $\tau\colon G\times G\to G$, $(g_1,g_2)\mapsto g_1g_2^{-1}$. Recalling that the geodesic symmetry s_o at $o:=\tau(\Delta)=e$ is given by $s_o\circ\tau=\tau\circ\sigma$ we obtain $s_o(g)=g^{-1}$ for $g\in G$.

Exercise. Using Lemma 1.9 prove that for any $h,g\in G$ we have $s_h(g)=hg^{-1}h$.

Next let $\mathfrak{g}$ denote the Lie algebra of G, and $e\colon\mathfrak{g}\to G$ the Lie group exponential mapping. Then the product algebra $\mathfrak{g}\times\mathfrak{g}$ is the Lie algebra of $G\times G$, and the identity

$$(X,Y)=\big(\tfrac{1}{2}(X+Y),\tfrac{1}{2}(X+Y)\big)+\big(\tfrac{1}{2}(X-Y),-\tfrac{1}{2}(X-Y)\big)$$

gives the Cartan decomposition of $\mathfrak{g}\times\mathfrak{g}$ into the two eigenspaces of $\Theta=D_{(e,e)}\sigma$. In particular, we have $\mathfrak{p}:=\{(X,-X):X\in\mathfrak{g}\}\subset\mathfrak{g}\times\mathfrak{g}$. So if $\hat{e}\colon\mathfrak{g}\times\mathfrak{g}\to G\times G$ denotes the Lie group exponential mapping, and $\exp\colon TG\to G$ the Riemannian exponential mapping of the symmetric space G, then (1.3) implies that for all $X\in\mathfrak{g}$,

$$\exp_o\big(D_{(e,e)}\tau(X,-X)\big)=\tau(\hat{e}^{(X,-X)})=\tau(e^X,e^{-X})=e^{2X}.$$

We conclude that the geodesics in the symmetric space G through the base point $o=e$ are the one-parameter subgroups of the group G.

1.6 The Killing form

For Lie groups the Killing form is an important and natural bilinear form on the Lie algebra. We will see that it also plays a very important role in the theory of globally symmetric spaces. In this section we will describe the Killing form and some of its properties. For that we need some more facts from the theory of Lie groups.

Recall the adjoint representation $\mathrm{Ad}:=\mathrm{Ad}_G\colon G\to\mathrm{GL}(\mathfrak{g})$ described in Section 1.3. The Lie algebra of $\mathrm{GL}(\mathfrak{g})$ is the vector space $\mathfrak{gl}(\mathfrak{g})$ of all linear endomorphisms of $\mathfrak{g}$ endowed with the bracket $[\Phi,\Psi]:=\Phi\circ\Psi-\Psi\circ\Phi$ for $\Phi,\Psi\in\mathfrak{gl}(\mathfrak{g})$. It is naturally identified with the tangent space of $\mathrm{GL}(\mathfrak{g})$ at the identity map $\mathrm{id}_{\mathfrak{g}}$. The *adjoint representation* $\mathrm{ad}\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{g}$ is defined as the differential $\mathrm{ad}:=D_e\,\mathrm{Ad}$ of the map $\mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$ at the identity of G. It can be shown that for $X\in\mathfrak{g}$ the endomorphism $\mathrm{ad}(X)$ is given by $\mathrm{ad}(X)Y=[X,Y]$ for $Y\in\mathfrak{g}$. Moreover we have the relation

$$\mathrm{Ad}(e^X)=e^{\mathrm{ad}(X)},\quad X\in\mathfrak{g},\tag{1.8}$$

where on the left-hand side $e\colon\mathfrak{g}\to G$ denotes the Lie group exponential mapping and on the right-hand side $e\colon\mathfrak{gl}(\mathfrak{g})\to\mathrm{GL}(\mathfrak{g})$ denotes the exponential mapping given by the usual power series

$$e^\Phi:=\mathrm{Id}+\Phi+\frac{1}{2}\Phi\circ\Phi+\sum_{k=3}^{\infty}\frac{1}{k!}\Phi^k,\quad \Phi\in\mathfrak{gl}(\mathfrak{g}).\tag{1.9}$$

The *Killing form* B of a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form on $\mathfrak{g}$ defined by

$$B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R},\ (X,Y)\mapsto \operatorname{Tr}(\operatorname{ad}(X)\circ\operatorname{ad}(Y)),$$

where $\operatorname{Tr}\colon \mathfrak{gl}(\mathfrak{g})\to\mathbb{R}$ is the canonical trace map. The following properties of the Killing form will turn out to be very useful:

Proposition 1.19 ([H], II.6 (2)). (1) $B(X,[Y,Z]) = B(Y,[Z,X]) = B(Z,[X,Y])$ *for any* $X,Y,Z\in\mathfrak{g}$.

(2) *For any Lie algebra automorphism* $\Phi\colon\mathfrak{g}\to\mathfrak{g}$ *we have* $B(\Phi(X),\Phi(Y)) = B(X,Y)$, $X,Y\in\mathfrak{g}$.

Definition 1.20. A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called *semi-simple* if its Killing form is non-degenerate. A Lie group is called semi-simple if its Lie algebra is.

So if G is a connected semi-simple Lie group, we can construct from the Killing form a natural G-invariant semi-Riemannian metric on the analytic manifold G as follows: requiring that the Riemannian exponential mapping $\exp_e : T_eG\to G$ at the identity e coincides with the Lie group exponential mapping $e\colon\mathfrak{g}\to G$ we get a natural identification of the tangent space T_eG at the identity $e\in G$ with the Lie algebra $\mathfrak{g}$ of G. Let Q_e be the non-degenerate symmetric bilinear form on T_eG corresponding to the Killing form of $\mathfrak{g}$. If for $g\in G$ the map $L_g\in\operatorname{Aut}(G)$ denotes left multiplication by g on the analytic manifold G, then its differential at a point $h\in G$ is a linear map $D_hL_g\colon T_hG\to T_{gh}G$. We define a non-degenerate symmetric bilinear form Q_g on T_gG via

$$Q_g(v,w) := Q_e\big((D_eL_g)^{-1}(v),(D_eL_g)^{-1}(w)\big),\quad v,w\in T_gG. \tag{1.10}$$

Doing this for all $g\in G$, we get a semi-Riemannian structure Q on G. Moreover, if $g,h\in G$ and $v,w\in T_gG$ are arbitrary, then using $D_eL_{hg} = D_gL_h\circ D_eL_g$ we compute

$$\begin{aligned}
&Q_{hg}(D_gL_h(v),D_gL_h(w))\\
&\quad\overset{(1.10)}{=} Q_e\big((D_eL_{hg})^{-1}(D_gL_h(v)),(D_eL_{hg})^{-1}(D_gL_h(w))\big)\\
&\quad= Q_e\big((D_eL_g)^{-1}(v),(D_eL_g)^{-1}(w)\big)\\
&\quad\overset{(1.10)}{=} Q_g(v,w).
\end{aligned}$$

So Q is indeed G-left-invariant.

For an arbitrary (not necessarily semi-simple) Lie algebra $\mathfrak{g}$ the following proposition will be very convenient in the sequel.

Proposition 1.21 ([H], Proposition II.6.8). *If* $\mathfrak{u}\subset\mathfrak{g}$ *is a compactly embedded subalgebra with* $\mathfrak{u}\cap\mathfrak{z}=\{0\}$, *then the Killing form* $B|_{\mathfrak{u}}$ *restricted to* $\mathfrak{u}$ *is negative definite.*

Notice that $\mathfrak{g}$ being semi-simple necessarily implies that the center $\mathfrak{z}$ of $\mathfrak{g}$ is trivial. So in this case the Killing form restricted to any compactly embedded subalgebra is negative definite.

For the remainder of this section we let S be a globally symmetric space with base point $o \in S$, (G, K) the associated Riemannian symmetric pair, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition with $e^{\mathfrak{k}} = K$. Then $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$, and $\mathfrak{k} \cap \mathfrak{z} = \{0\}$ by the last assertion of Theorem 1.7. So from the previous proposition we know that the Killing form restricted to $\mathfrak{k}$ is negative definite.

We will now prove several useful lemmata.

Lemma 1.22. *$\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form.*

Proof. Let $Z \in \mathfrak{k}$, $X \in \mathfrak{p}$ arbitrary, and $\Theta\colon \mathfrak{g} \to \mathfrak{g}$ the Cartan involution. By definition of $\mathfrak{k}$ and $\mathfrak{p}$ we have $\Theta(Z) = Z$ and $\Theta(X) = -X$. Moreover, since Θ is a Lie algebra automorphism, we have from Proposition 1.19 (2)

$$B(Z, X) = B(\Theta(Z), \Theta(X)) = B(Z, -X) = -B(Z, X),$$

which implies $B(Z, X) = 0$. $\square$

Lemma 1.23. *For all $k \in K$ we have* $\mathrm{Ad}(k)\mathfrak{p} = \mathfrak{p}$.

Proof. Let $Z \in \mathfrak{k}$ be such that $k = e^Z$, and $X \in \mathfrak{p}$ be arbitrary. Then

$$\mathrm{Ad}(k)X = \mathrm{Ad}(e^Z)X \overset{(1.8)}{=} e^{\mathrm{ad}(Z)}X \overset{(1.9)}{=} X + \mathrm{ad}(Z)X + \sum_{i=2}^{\infty} \frac{1}{i!}\,\mathrm{ad}(Z)^i X.$$

Since $\mathrm{ad}(Z)X = [Z, X] \in \mathfrak{p}$, and inductively $\mathrm{ad}(Z)^i X = \mathrm{ad}(Z)^{i-1}[Z, X] \in \mathfrak{p}$ for $i \geq 2$ by Lemma 1.14, we get $\mathrm{Ad}(k)\mathfrak{p} \subseteq \mathfrak{p}$.

The reverse inclusion follows from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the fact that $\mathrm{Ad}(k)\colon \mathfrak{k} \to \mathfrak{k}$ and $\mathrm{Ad}(k)\colon \mathfrak{g} \to \mathfrak{g}$ are isomorphisms. $\square$

For any $x \in S$ we denote by $\tau_x\colon G \to S$, $g \mapsto g \cdot x$, the orbit map, and $D_e\tau_x\colon \mathfrak{g} \to T_x S$ its differential at the identity $e \in G$. We know from Theorem 1.15 that $D_e\tau_o$ maps $\mathfrak{p}$ isomorphically into $T_o S$. For $x \in S$ arbitrary, we have the following:

Lemma 1.24. *If $x \in S$ and $g \in G$ such that $g \cdot o = x$, then $D_e\tau_x$ maps* $\mathrm{Ad}(g)\mathfrak{p}$ *isomorphically into $T_x S$.*

Proof. For $g \in G = \mathrm{Is}^o(S)$ we denote by $D_o g\colon T_o S \to T_{g\cdot o} S$ its differential at the base point $o \in S$, and L_g, $R_g \in \mathrm{Aut}(G)$ left- and right-multiplication by g on G. Now let $x \in S$ and $g \in G$ be such that $g \cdot o = x$. From the definitions we immediately get

$$\tau_o \circ L_g = g \circ \tau_o, \quad \tau_x = \tau_o \circ R_g.$$

If $X \in \mathfrak{g}$ then by the above relations we have

$$\tau_x(e^{tX}) = \tau_o \circ R_g(e^{tX}) = \underbrace{\tau_o \circ L_g}_{=g\circ\tau_o} \circ (L_g)^{-1} \circ R_g(e^{tX}),$$

and from $(L_g)^{-1} \circ R_g(e^{tX}) = g^{-1}e^{tX}g = e^{t\,\mathrm{Ad}(g^{-1})X}$ we conclude

$$D_e\tau_x(X) = \frac{d}{dt}\Big|_{t=0}\tau_x(e^{tX}) = \frac{d}{dt}\Big|_{t=0} g\cdot\tau_o\big(e^{t\,\mathrm{Ad}(g^{-1})X}\big) = D_o g \circ D_e\tau_o\big(\mathrm{Ad}(g^{-1})X\big).$$

Since $D_o g$ is an isomorphism we therefore have $X \in \ker\big(D_e\tau_x\big)$ if and only if $\mathrm{Ad}(g^{-1})X \in \ker\big(D_e\tau_o\big) = \mathfrak{k}$. This is equivalent to $\ker\big(D_e\tau_x\big) = \mathrm{Ad}(g)\mathfrak{k}$.

We know from Lemma 1.22 that $\mathfrak{g}$ is the orthogonal direct sum of $\mathfrak{k}$ and $\mathfrak{p}$. Since $\mathrm{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism and hence by Proposition 1.19 (2) preserves the Killing form, we know that $\mathfrak{g} = \mathrm{Ad}(g)\mathfrak{g}$ can be decomposed into the orthogonal direct sum $\mathrm{Ad}(g)\mathfrak{k} \oplus \mathrm{Ad}(g)\mathfrak{p}$. Hence $D_e\tau_x\big|_{\mathrm{Ad}(g)\mathfrak{p}}$ is an isomorphism. $\square$

Notice that if $g, h \in G$ satisfy $g \cdot o = h \cdot o = x$, then $h^{-1}g$ fixes o and therefore belongs to K, so by Lemma 1.23

$$\mathrm{Ad}(g)\mathfrak{p} = \mathrm{Ad}(h)\underbrace{\big(\mathrm{Ad}(h^{-1}g)\mathfrak{p}\big)}_{=\mathfrak{p}} = \mathrm{Ad}(h)\mathfrak{p}.$$

This shows that the map $D_o\tau_x$ does not depend on the choice of $g \in G$ such that $g \cdot o = x$. Moreover, the decomposition $\mathfrak{g} = \mathrm{Ad}(g)\mathfrak{k} \oplus \mathrm{Ad}(g)\mathfrak{p}$ can be interpreted as the Cartan decomposition of $\mathfrak{g}$ with respect to the involution induced by the geodesic symmetry s_x at $x = g\cdot o \in S$; the isotropy subgroup of G at x is the compact subgroup $e^{\mathrm{Ad}(g)\mathfrak{k}} = gKg^{-1}$.

1.7 Decomposition of symmetric spaces

We have seen in Section 1.3 that a globally symmetric space together with the choice of a base point $o \in S$ gives rise to a pair $(\mathfrak{g}, \Theta)$, where $\mathfrak{g}$ is the Lie algebra of the group of isometries $\mathrm{Is}^o(S)$, and Θ the differential at the identity of the involutive automorphism σ of G induced by the geodesic symmetry at o. Moreover, the set of fixed points of Θ in $\mathfrak{g}$ is a compactly embedded subalgebra. In this section we will have a look at such pairs.

Definition 1.25. An *orthogonal symmetric Lie algebra* is a pair $(\mathfrak{l}, \varsigma)$, where $\mathfrak{l}$ is a Lie algebra over $\mathbb{R}$ and ς is an involutive automorphism of $\mathfrak{l}$ such that $\mathfrak{u} = \{X \in \mathfrak{l} : \varsigma X = X\}$ is a compactly embedded subalgebra of $\mathfrak{l}$.

$(\mathfrak{l}, \varsigma)$ is called *effective* if in addition $\mathfrak{u} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z} \subset \mathfrak{l}$ denotes the center of $\mathfrak{l}$.

Notice that any pair $(\mathfrak{g}, \Theta)$ coming from a globally symmetric space is effective by the last assertion of Theorem 1.7.

Definition 1.26. Let $(\mathfrak{l}, \varsigma)$ be an effective orthogonal symmetric Lie algebra with Killing form B, and $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$ the decomposition of $\mathfrak{l}$ into the eigenspaces of ς for the eigenvalue $+1$ and -1 respectively. Then $(\mathfrak{l}, \varsigma)$ is said to be of

(1) *compact type* if $\mathfrak{l}$ is compact and semi-simple;

(2) *non-compact type* if $\mathfrak{l}$ is non-compact and semi-simple, and if $B|_{\mathfrak{u}}$ is negative definite and $B|_{\mathfrak{e}}$ is positive definite;

(3) *Euclidean type* if $\mathfrak{e}$ is an abelian ideal in $\mathfrak{l}$.

Notice that the proof of Lemma 1.22 shows that the subspaces $\mathfrak{u}$ and $\mathfrak{e}$ are orthogonal with respect to the Killing form. Moreover, Proposition 1.21 implies that the Killing form restricted to $\mathfrak{u}$ is negative definite.

We say that a pair (L, U) is associated with an orthogonal symmetric Lie algebra $(\mathfrak{l}, \varsigma)$ if L is a connected Lie group with Lie algebra $\mathfrak{l}$, and U is a Lie subgroup of L with Lie algebra $\mathfrak{u}$. So we can define the type of a pair (L, U) according to the type of the effective orthogonal Lie algebra it is associated to. Similarly, the type of a globally symmetric space S is defined as the type of an associated Riemannian symmetric pair (G, K) (which is naturally associated to an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ as above). Notice that even though every choice of base point a priori gives rise to a different Riemannian symmetric pair, the types of all such pairs are the same: if instead of a base point $o \in S$ we take the base point $x = g \cdot o$, $g \in G$, then the Lie algebra $\mathfrak{g}$ remains the same and only the involution Θ on $\mathfrak{g}$ is changed to $\mathrm{Ad}(g)\Theta$.

Example 1. $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ is a symmetric space of non-compact type: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ is non-compact and semi-simple. Moreover, $B|_{\mathfrak{k}}$ is negative definite by Proposition 1.21, and $B|_{\mathfrak{p}}$ is a positive multiple of the positive definite symmetric bilinear form (1.4).

Example 2. If $G \subset \mathrm{SL}(n, \mathbb{R})$ is a closed subgroup invariant under transposition and $K = \mathrm{SO}(n) \cap G$, then G/K is also a symmetric space of non-compact type, because $\mathfrak{g}$ is non-compact and semi-simple, $B|_{\mathfrak{k}}$ is negative definite by Proposition 1.21, and $B|_{\mathfrak{p}}$ is a positive multiple of the positive definite symmetric bilinear form (1.4).

Example 3. $\mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$ is a symmetric space of compact type: $\mathfrak{g} = \mathfrak{so}(p+q)$ is compact and semi-simple. Notice that in this case $B|_{\mathfrak{p}}$ is a negative multiple of the positive definite symmetric bilinear form (1.7).

Example 4. A compact connected semi-simple Lie group $G \cong (G \times G)/\Delta$ is a symmetric space of compact type with respect to any metric induced by a G-bi-invariant metric on G. The corresponding orthogonal symmetric Lie algebra is $(\mathfrak{g} \times \mathfrak{g}, \Theta)$, where $\Theta(X, Y) := (Y, X)$ for $X, Y \in \mathfrak{g}$, and $\mathfrak{g} \times \mathfrak{g}$ is compact and semi-simple.

The next theorem gives a decomposition for effective orthogonal symmetric Lie algebras.

Theorem 1.27 ([H], Theorem V.1.1). *Let $(\mathfrak{l}, \varsigma)$ be an effective orthogonal symmetric Lie algebra. Then there exist ideals l_0, l_- and l_+ such that*

(1) *$\mathfrak{l}$ can be decomposed as a direct sum $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_- \oplus \mathfrak{l}_+$;*

(2) *$\mathfrak{l}_0$, $\mathfrak{l}_-$ and $\mathfrak{l}_+$ are invariant under ς and orthogonal with respect to the Killing form of $\mathfrak{l}$;*

(3) *the pairs $(\mathfrak{l}_0, \varsigma|_{\mathfrak{l}_0})$, $(\mathfrak{l}_-, \varsigma|_{\mathfrak{l}_-})$ and $(\mathfrak{l}_+, \varsigma|_{\mathfrak{l}_+})$ are effective orthogonal symmetric Lie algebras of Euclidean type, compact type and non-compact type respectively.*

Let (L, U) be a pair associated with an effective orthogonal symmetric Lie algebra $(\mathfrak{l}, \varsigma)$, and $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$ the decomposition of $\mathfrak{l}$ into the eigenspaces of ς for the eigenvalues $+1$ and -1 respectively. In the proof of the above theorem, S. Helgason shows that for any $\mathrm{Ad}(U)$-invariant positive definite symmetric bilinear form Q on $\mathfrak{e}$ there exists an endomorphism Φ of $\mathfrak{e}$ such that

$$Q(\Phi(X), Y) = B(X, Y) \quad \text{for any } X, Y \in \mathfrak{e}.$$

Moreover, if $\mathfrak{l}$ is of Euclidean, compact or non-compact type, then all eigenvalues of Φ are identically zero, strictly negative or strictly positive, respectively. This immediately implies the following

Proposition 1.28. *If $(\mathfrak{l}_0, \varsigma_0)$, $(\mathfrak{l}_-, \varsigma_-)$, $(\mathfrak{l}_+, \varsigma_+)$ are effective orthogonal symmetric Lie algebras of Euclidean, compact and non-compact type respectively, then the Killing form restricted to $\mathfrak{e}_0$, $\mathfrak{e}_-$, $\mathfrak{e}_+$ is identically zero, negative definite and positive definite, respectively.*

For the remainder of this section we let S be a globally symmetric space, $o \in S$ a base point, and (G, K) the associated Riemannian symmetric pair. Let $\tau\colon G \to S$, $g \mapsto g \cdot o$, denote the natural map, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with $e^{\mathfrak{k}} = K$. From the previous proposition we know that if S is of compact type, then $-B|_{\mathfrak{p}}$ induces a scalar product Q_o on T_oS, and if S is of non-compact type, then $B|_{\mathfrak{p}}$ does.

As performed in Section 1.6 for a semi-simple Lie group, we can extend this scalar product to a G-invariant Riemannian structure on S: for $g \in G$ we denote by $Dg\colon TS \to TS$ the differential of the isometry g. For $x \in S$ we choose $g \in G$ such that $x = g \cdot o$ and define a scalar product Q_x on T_xS via

$$Q_x(v, w) := Q_o\big((D_og)^{-1}(v), (D_og)^{-1}(w)\big), \quad v, w \in T_xS. \tag{1.11}$$

Notice that if $h \in G$ also satisfies $h \cdot o = x$, then $h^{-1}g \in K$. Moreover, for any $k \in K$ the Killing form $B_{\mathfrak{p}}$ restricted to $\mathfrak{p}$ is $\mathrm{Ad}(k)$-invariant. Since $Q_o(D_e\tau(X), D_e\tau(Y)) = \mp B(X, Y)$, and $D_e\tau \circ \mathrm{Ad}(k) = D_ok \circ D_e\tau$ we conclude that Q_o is invariant under D_ok for any $k \in K$. Hence the assignment $x \mapsto Q_x$ is consistent and defines a Riemannian structure Q on S. This structure is G-invariant: indeed, if $x, y \in S$, $v, w \in T_xS$ are arbitrary, and $g \in G$ such that $y = g \cdot x$, $h \in G$ such that $x = h \cdot o$, then $y = gh \cdot o$ and $D_o(gh) = D_x g \circ D_o h$ gives

$$Q_y(D_x g(v), D_x g(w)) \overset{(1.11)}{=} Q_e\big((D_o(gh))^{-1} \circ D_x g(v), (D_o(gh))^{-1} \circ D_x g(w)\big)$$
$$= Q_e\big((D_o h)^{-1}(v), (D_o h)^{-1}(w)\big) \overset{(1.11)}{=} Q_x(v, w).$$

Hence both for symmetric spaces S of compact type and of non-compact type the Killing form canonically induces a G-invariant Riemannian structure on S.

Conversely, any G-invariant Riemannian metric Q on a symmetric space S of compact or non-compact type is essentially determined by the restriction of the Killing form to $\mathfrak{p}$: since $Q_o \circ D_e\tau|_{\mathfrak{p}}$ is an $\mathrm{Ad}(K)$-invariant positive definite symmetric bilinear form on $\mathfrak{p}$, by the remark following Theorem 1.27 there exists an automorphism Φ of $\mathfrak{p}$ such that in $T_o S = D_e\tau(\mathfrak{p})$ we have

$$Q_o(D_e\tau(\Phi(X)), D_e\tau(Y)) = B(X, Y) \quad \text{for all } X, Y \in \mathfrak{p}.$$

Moreover, according to whether S is of compact type or of non-compact type, all eigenvalues of Φ are strictly negative or strictly positive, and all eigenspaces of Φ are invariant by $\mathrm{Ad}(K)$. More details can be found e.g. in Section 2.3 of [E], Section 8.2 of [Wo] or Chapter V, §1 and §3 in [H].

We will now look at the sectional curvature of the globally symmetric space S. Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathfrak{p} \cong T_o S$ induced from the Riemannian metric. Recall from Theorem 1.17 that for $X, Y, Z \in \mathfrak{p}$ the curvature tensor $R(X, Y)Z := (D_e\tau)^{-1}\big(R_o(D_e\tau(X), D_e\tau(Y))D_e\tau(Z)\big)$ is given by $-[[X, Y], Z]$. Given two linearly independent vectors $X, Y \in \mathfrak{p}$, the sectional curvature $\kappa(\langle X, Y\rangle)$ of the two-plane $\langle X, Y\rangle$ in $T_o S$ spanned by $D_e\tau(X)$ and $D_e\tau(Y)$ is defined by

$$\kappa(\langle X, Y\rangle) := \frac{\langle R(Y, X)X, Y\rangle}{\sqrt{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}}.$$

We have the following

Theorem 1.29 ([H], Theorem V.3.1). *Let S be a globally symmetric space with associated Riemannian symmetric pair (G, K) such that $K \subset G$ is connected and closed, and Q an arbitrary G-invariant Riemannian metric.*

(1) *If S is of compact type, then S has non-negative sectional curvature.*

(2) *If S is of non-compact type, then S has non-positive sectional curvature.*

(3) *If S is of Euclidean type, then the sectional curvature of S is identically zero.*

Proof. Notice that by G-invariance of the metric it suffices to prove the claim for arbitrary two-planes in $T_o S \cong \mathfrak{p}$. Recall that $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathfrak{p}$ induced by the Riemannian structure Q. We first prove (1) and (2). By the remark following Proposition 1.28 there exists an automorphism Φ of $\mathfrak{p}$ with all eigenvalues strictly positive such that $\langle \Phi(X), Y\rangle = \mp B(X, Y)$, $X, Y \in \mathfrak{p}$, according to whether S is of compact type or of non-compact type.

Now choose an arbitrary two-plane E in T_oS and a basis $X, Y \in \mathfrak{p}$ of $(D_e\tau)^{-1}(E)$ satisfying $\langle X, X\rangle = \langle Y, Y\rangle = 1$ and $\langle X, Y\rangle = 0$. Then the sectional curvature of E is given by

$$\kappa(E) = \langle R(Y, X)X, Y\rangle = \langle -[[Y, X], X], Y\rangle = -\langle Y, [[Y, X], X]\rangle. \tag{1.12}$$

Let $k \in K$ be such that $\mathrm{Ad}(k)Y$ is an eigenvector of Φ with eigenvalue say $\beta > 0$. Since the scalar product is invariant under $\mathrm{Ad}(k)$ we get

$$\kappa(E) = -\langle \mathrm{Ad}(k)Y, \mathrm{Ad}(k)[[Y, X], X]\rangle = -\beta^{-1}\langle \Phi(\mathrm{Ad}(k)Y), \mathrm{Ad}(k)[[Y, X], X]\rangle.$$

If S is of compact type, we therefore have by $\mathrm{Ad}(k)$-invariance of the Killing form and Proposition 1.19 (1)

$$\begin{aligned}\beta\,\kappa(E) &= B(\mathrm{Ad}(k)Y, \mathrm{Ad}(k)[[Y, X], X]) = B(Y, [[Y, X], X])\\ &= B([Y, X], [X, Y]) = -B([X, Y], [X, Y]) \geq 0,\end{aligned}$$

because B is negative definite on $\mathfrak{k}$ and $[X, Y] \in \mathfrak{k}$. Similarly, for S of non-compact type we get $\beta\,\kappa(E) = B([X, Y], [X, Y]) \leq 0$. The claim then follows from the fact that all eigenvalues of Φ are positive.

If S is of Euclidean type, then $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$. So for all $X, Y \in \mathfrak{p}$ we have $R(Y, X)X = -[[Y, X], X] = 0$, hence by (1.12) $\kappa(E) = 0$ for any two-plane $E \subseteq T_oS$. $\square$

Notice that for the non-compact type, the hypothesis that K is connected and closed is always satisfied; for the compact type, K is always closed but not necessarily connected.

We finally state the de Rham decomposition

Theorem 1.30 ([H], Proposition V.4.2). *Let S be a globally symmetric space. Then*

$$S = S_0 \times S_- \times S_+,$$

where S_0 is a symmetric space of Euclidean type, S_- a symmetric space of compact type, and S_+ a symmetric space of non-compact type.

This theorem implies that in order to understand arbitrary globally symmetric spaces it suffices to study symmetric spaces of Euclidean, compact and non-compact type separately. Since the symmetric spaces of Euclidean type are isometric to Euclidean spaces by Theorem 1.29 (3), the interesting classes of symmetric spaces are those of compact or of non-compact type.

2 Symmetric spaces of non-compact type

In this section we will study the structure of globally symmetric spaces of non-compact type which are known to be non-positively curved by Theorem 1.29. Moreover, it follows from Definition 1.26 that the connected component of the identity of the isometry group is a semi-simple Lie group. More precisely, we have the following

Proposition 2.1 ([E], Proposition 2.1.1). *The connected component of the identity of the isometry group of a globally symmetric space of non-compact type is a semi-simple Lie group with trivial center and without compact factor.*

We will explain the classical decompositions of semi-simple Lie groups and relate them to the geometry of the symmetric space. In this way we can understand how totally geodesically embedded Euclidean spaces and the so-called horocycles sit inside our manifold.

For the remainder of this text we will assume that S is a globally symmetric space of non-compact type, $o \in S$ a base point, $G = \mathrm{Is}^o(S)$ and $K \subset G$ the compact isotropy subgroup at o. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, and $D_e\tau \colon \mathfrak{p} \to T_o S$ the isomorphism given in Theorem 1.15. We will assume that the Riemannian structure of S is induced by the Killing form $B|_{\mathfrak{p}}$ restricted to $\mathfrak{p}$; from the remark following Proposition 1.28 we know that this is not a severe restriction.

2.1 Flats and rank

Definition 2.2. A k*-flat* in S is a totally geodesic k-dimensional submanifold isometric to $\mathbb{R}^k$. The *rank* r *of* S is defined as the maximal natural number r for which an r-flat exists in S. An r-flat is called a (maximal) flat.

Notice that a 1-flat is simply a geodesic. Moreover, if a symmetric space of non-compact type has an upper negative bound on its sectional curvature, then it is of rank one. The rank one symmetric spaces of non-compact type are completely classified: they are precisely the hyperbolic spaces over the reals, complex numbers and quaternions, and the hyperbolic plane over the Cayley numbers. Every other symmetric space of non-compact type is of rank bigger than one and therefore possesses totally geodesically embedded Euclidean planes.

We next address the following question: how does an r-flat F through the base point $o \in S$ look like in terms of Lie algebras?

We first remark that F is totally geodesic, hence by Theorem 1.17 (2) it necessarily has the form $F = e^{\mathfrak{q}} \cdot o$ for a Lie triple system $\mathfrak{q} \subseteq \mathfrak{p}$. Moreover, the sectional curvature restricted to F equals zero. Hence for all $X, Y \in \mathfrak{q} \cong T_o F$ such that $B(X,Y) = 0$, $B(X,X) = B(Y,Y) = 1$ we have the condition

$$0 = \kappa(\langle X, Y \rangle) = B([X,Y],[X,Y]).$$

Again, the fact that B is negative definite on $\mathfrak{k}$ implies that $[X,Y] = 0$. Hence $\mathfrak{q} \subseteq \mathfrak{p}$ has to be an abelian subspace.

Now let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace of dimension $r = \mathrm{rank}(S)$. Then $F = e^{\mathfrak{a}} \cdot o$ is a maximal flat in S. Since G acts by isometries on S, every set of the form $g \cdot F$, $g \in G$, is also a maximal flat. We will see later on that every flat in S is necessarily a G-translate of F.

Example 1. For $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ we know from Section 1.5 that $\mathfrak{p} = \mathrm{sym}_0(n) \subset \mathfrak{sl}(n,\mathbb{R})$, the set of symmetric $(n \times n)$-matrices of trace zero with entries in $\mathbb{R}$. A maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is the set of diagonal matrices of trace zero, i.e.

$$\mathfrak{a} = \big\{ \mathrm{Diag}(t_1, \dots, t_n) : t_1, \dots, t_n \in \mathbb{R},\ \textstyle\sum_{i=1}^n t_i = 0 \big\}.$$

We have seen that the set of positive definite symmetric $(n \times n)$-matrices $\mathrm{Pos}_1(n)$ with determinant one is diffeomorphic to $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, where the $\mathrm{SL}(n,\mathbb{R})$-action on $\mathrm{Pos}_1(n)$ is given by $g \cdot p := g^t p g$, $p \in \mathrm{Pos}_1(n)$, $g \in \mathrm{SL}(n,\mathbb{R})$. The base point $o \in \mathrm{Pos}_1(n)$ is the fixed point of $\mathrm{SO}(n)$, hence the identity matrix $I_n \in \mathrm{Pos}_1(n)$. So we get a maximal flat

$$\begin{aligned} F = e^{\mathfrak{a}} \cdot o &= \big\{ \mathrm{Diag}(e^{2t_1}, \dots, e^{2t_n}) : t_1, \dots, t_n \in \mathbb{R},\ \textstyle\sum_{i=1}^n t_i = 0 \big\} \\ &= \big\{ \mathrm{Diag}(\lambda_1, \dots, \lambda_n) : \lambda_1, \dots, \lambda_n > 0,\ \textstyle\prod_{i=1}^n \lambda_i = 1 \big\} \end{aligned} \tag{2.1}$$

in S, and the rank equals $n-1$.

Exercise. Every flat F in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is isomorphic to the Euclidean vector space $\mathbb{R}^{n-1}$, hence for $x, y, z \in F$ the angle $\angle_x(y,z)$ between the vectors pointing from x to y and from x to z is well-defined. Using formula (1.4), show that for x, y, z in a common flat F with $\angle_x(y,z) = \pi/2$ we have $d(y,z)^2 = d(x,y)^2 + d(x,z)^2$.

Example 2a. For $\mathrm{SO}(p,q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$, $p \le q$, a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \big\{ \left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right) : D = (d_{ij}) \in M(p,q),\ d_{ij} = 0 \text{ for } i \neq j \big\}.$$

In particular, the rank equals $p = \min\{p,q\}$.

Example 2b. For $\mathrm{Sp}(2q,\mathbb{R})/(\mathrm{SO}(2q) \cap \mathrm{Sp}(2q,\mathbb{R}))$ a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \big\{ \left(\begin{smallmatrix} D & 0 \\ 0 & -D \end{smallmatrix}\right) : D = \mathrm{Diag}(t_1, \dots, t_q),\ t_1, \dots, t_q \in \mathbb{R} \big\}.$$

In particular, the rank equals q.

As in Section 1.5 we consider the set S_{2q} of ω-compatible complex structures on the symplectic vector space $(\mathbb{R}^{2q}, \omega)$ with the $\mathrm{Sp}(2q,\mathbb{R})$-action by conjugation. Choosing as a base point $o \in \mathrm{Sp}(2q,\mathbb{R})$ the ω-compatible complex structure defined by the matrix J_0 given in (1.6), S_{2q} is diffeomorphic to $\mathrm{Sp}(2q,\mathbb{R})/(\mathrm{SO}(2q) \cap \mathrm{Sp}(2q,\mathbb{R}))$. The following set F is a maximal flat in S_{2q}:

$$\begin{aligned} F = e^{\mathfrak{a}} \cdot o &= \big\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right) \begin{pmatrix} 0 & -I_q \\ I_q & 0 \end{pmatrix} \left(\begin{smallmatrix} A^{-1} & 0 \\ 0 & A \end{smallmatrix}\right) : A = \mathrm{Diag}(e^{t_1}, \dots, e^{t_q}),\ t_1, \dots, t_q \in \mathbb{R} \big\} \\ &= \big\{ \begin{pmatrix} 0 & -A^2 \\ A^{-2} & 0 \end{pmatrix} : A = \mathrm{Diag}(e^{t_1}, \dots, e^{t_q}),\ t_1, \dots, t_q \in \mathbb{R} \big\} \\ &= \big\{ \begin{pmatrix} 0 & \mathrm{Diag}(-\lambda_1, \dots, -\lambda_q) \\ \mathrm{Diag}(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_q}) & 0 \end{pmatrix} : \lambda_1, \dots, \lambda_q > 0 \big\}. \end{aligned}$$

Example 2c. A maximal flat in $\mathbb{H}^2 \times \mathbb{H}^2$ is simply a set

$$\{(c_1(t_1), c_2(t_2)) : t_1, t_2 \in \mathbb{R}\},$$

where c_i is a geodesics in the i-th $\mathbb{H}^2$-factor for $i = 1, 2$.

Lemma 2.3. *Every geodesic is contained in at least one flat.*

Proof. If $c \subset S$ is a geodesic, then by the remark following Corollary 1.16 there exists $g \in G$ and $X \in \mathfrak{p}$ such that $c(t) = ge^{tX} \cdot o$, $t \in \mathbb{R}$. Take a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ which contains X. Then c is contained in the flat $ge^{\mathfrak{a}} \cdot o$. $\square$

Definition 2.4. Let $X \in \mathfrak{p}$, and $Z_{\mathfrak{g}}(X) := \{Y \in \mathfrak{g} : [Y, X] = 0\}$ the centralizer of X in $\mathfrak{g}$. The vector X is called *regular* if $Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian, and *singular* otherwise.

Notice that if $X \in \mathfrak{p}$ is singular, then $\dim(Z_{\mathfrak{g}}(X) \cap \mathfrak{p}) > r$. The following lemma in particular shows that regular vectors exist.

Lemma 2.5 ([H], Lemma V.6.3 (i)). *Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Then there exists an element $H \in \mathfrak{a}$ such that $Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a}$.*

Theorem 2.6. *If $\mathfrak{a}$, $\mathfrak{a}'$ are maximal abelian subspaces of $\mathfrak{p}$, then there exists $k \in K$ such that $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$.*

Proof. Choose $H \in \mathfrak{a}$, $H' \in \mathfrak{a}'$ regular. Recall that B denotes the Killing form on $\mathfrak{g}$, and consider the bounded differentiable map $f : K \to \mathbb{R}$ defined by $f(k) = B\big(\mathrm{Ad}(k)H, H'\big)$. Let $k_0 \in K$ be one of its critical points. Then for any $Z \in \mathfrak{k}$ we have

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} f(k_0 e^{tZ}) = \frac{d}{dt}\Big|_{t=0} B(\mathrm{Ad}(k_0 e^{tZ})H, H') \\
&= \frac{d}{dt}\Big|_{t=0} B(\mathrm{Ad}(k_0)\,\mathrm{Ad}(e^{tZ})H, H') = B(\mathrm{Ad}(k_0)\frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(e^{tZ})H), H') \\
&= B(\mathrm{Ad}(k_0)(\mathrm{ad}\, Z)H, H') = B(\mathrm{Ad}(k_0)[Z, H], H') \\
&= B(\mathrm{Ad}(k_0)Z, [\mathrm{Ad}(k_0)H, H']).
\end{aligned}$$

From Lemma 1.23 we know that $\mathrm{Ad}(k_0)H \in \mathfrak{p}$, so by Lemma 1.14, $[\mathrm{Ad}(k_0)H, H'] \in \mathfrak{k}$. Since both $\mathrm{Ad}(k_0)Z$ and $[\mathrm{Ad}(k_0)H, H']$ belong to $\mathfrak{k}$, $Z \in \mathfrak{k}$ is arbitrary, and the restriction of the Killing form B to $\mathfrak{k}$ is negative definite by Proposition 1.21, we conclude that $[\mathrm{Ad}(k_0)H, H'] = 0$. Since H' is regular, every element in $\mathfrak{g}$ which commutes with H' is contained in $\mathfrak{a}'$, so $\mathrm{Ad}(k_0)H \in \mathfrak{a}'$. Since $\mathfrak{a}'$ is abelian, every element in $\mathfrak{a}'$ commutes with $\mathrm{Ad}(k_0)H$, and therefore every element in $\mathrm{Ad}(k_0^{-1})\mathfrak{a}'$ commutes with H. Since $\mathfrak{a} = Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$ we conclude that $\mathrm{Ad}(k_0^{-1})\mathfrak{a}' \subseteq \mathfrak{a}$. Exchanging the role of $\mathfrak{a}$ and $\mathfrak{a}'$ in the argument above we conclude that there exists $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} \subseteq \mathfrak{a}'$. Hence

$$\mathrm{Ad}(k)\,\mathrm{Ad}(k_0^{-1})\mathfrak{a}' \subseteq \mathrm{Ad}(k)\mathfrak{a} \subseteq \mathfrak{a}',$$

which shows that $\mathrm{Ad}(k)\,\mathrm{Ad}(k_0^{-1})\mathfrak{a}' = \mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$. □

Notice that this theorem in particular implies that all maximal abelian subspaces of $\mathfrak{p}$ have the same dimension $r = \mathrm{rank}(S)$.

Lemma 2.7. *The vector $X \in \mathfrak{p}$ is regular if and only if the geodesic $c \subset S$ defined by $c(t) := e^{tX} \cdot o$, $t \in \mathbb{R}$, is contained in precisely one flat.*

Proof. Suppose $X \in \mathfrak{p}$ is regular, i.e. $\mathfrak{a} := Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian, and c is contained in more than one flat. Since every flat through the base point o is of the form $e^{\mathfrak{a}'} \cdot o$ with $\mathfrak{a}' \subset \mathfrak{p}$ a maximal abelian subspace, we may assume that $c \subseteq e^{\mathfrak{a}} \cdot o$ and $c \subseteq e^{\mathfrak{a}'} \cdot o$, $\mathfrak{a}' \subset \mathfrak{p}$ maximal abelian, $\mathfrak{a}' \neq \mathfrak{a}$. Since $X \in \mathfrak{a}'$ and $\mathfrak{a}'$ is abelian we conclude that every element in $\mathfrak{a}' \subset \mathfrak{p}$ commutes with X, hence is contained in $Z_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$. But $\mathfrak{a}' \subseteq \mathfrak{a}$ and $\dim \mathfrak{a}' = \dim \mathfrak{a} = \mathrm{rank}(S)$ then imply $\mathfrak{a}' = \mathfrak{a}$, a contradiction.

Conversely assume that c is contained in precisely one flat, say $e^{\mathfrak{a}} \cdot o$ for $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian. Suppose $Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is not maximal abelian. Then $\mathfrak{a} \subset Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ and there exists $X' \in Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$ such that $X' \notin \mathfrak{a}$. Choose $\mathfrak{a}' \subset \mathfrak{p}$ maximal abelian such that $X' \in \mathfrak{a}'$. Then by the choice of X' we have $\mathfrak{a}' \neq \mathfrak{a}$, and $X' \in Z_{\mathfrak{g}}(X)$ implies $[X', X] = 0$, hence $X \in \mathfrak{a}'$. We conclude that c is contained in the two different flats $e^{\mathfrak{a}} \cdot o$ and $e^{\mathfrak{a}'} \cdot o$, a contradiction. □

Example 1. $X \in \mathfrak{p}$ is regular if and only if all its eigenvalues are distinct.

Let us look in the case $n = 3$ at the singular vector $X = \mathrm{Diag}(1, 1, -2) \in \mathfrak{p} = \mathrm{sym}_0(3)$. An easy computation shows that for any $\theta \in \mathbb{R}$ the element $k(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K = \mathrm{SO}(3)$ satisfies $\mathrm{Ad}(k(\theta))X = X$.

In particular, there exists a one-parameter family of flats containing the geodesic c defined by $c(t) = e^{tX} \cdot o$, $t \in \mathbb{R}$.

Example 2b. $X \in \mathfrak{p}$ is regular if and only if all its eigenvalues are distinct and different from zero.

In the case $q = 2$ the vector $X = \mathrm{Diag}(1, 1, -1, -1) \in \mathfrak{p}$ is singular: clearly every element in K of the form

$$k = \left(\begin{array}{cc|cc} \cos\theta & \sin\theta & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\sin\theta & \cos\theta & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{array}\right), \quad \theta \in \mathbb{R},$$

satisfies $\mathrm{Ad}(k)X = X$. For $\mu_1, \mu_2 > 0$ we set

$$A_\theta(\mu_1, \mu_2) := \begin{pmatrix} \mu_1 \cos^2\theta + \mu_2 \sin^2\theta & (\mu_2 - \mu_1)\sin\theta\cos\theta \\ (\mu_2 - \mu_1)\sin\theta\cos\theta & \mu_1 \sin^2\theta + \mu_2 \cos^2\theta \end{pmatrix}.$$

Then the geodesic $c(t) := e^{tX} \cdot o = \begin{pmatrix} 0 & -e^t I_2 \\ e^{-t} I_2 & 0 \end{pmatrix} \subset S_4$ belongs to each of the following flats parametrized by $\theta \in \mathbb{R}$:

$$\left\{ \begin{pmatrix} 0 & A_\theta(-\lambda_1, -\lambda_2) \\ A_\theta(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}) & 0 \end{pmatrix} : \lambda_1, \lambda_2 > 0 \right\} \subset S_4.$$

Similarly, the vector $Y = \mathrm{Diag}(1, 0, -1, 0)$ is invariant by $\mathrm{Ad}(k)$ for any $k \in K$ of the form

$$k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

So for all $\theta \in \mathbb{R}$ the geodesic $c(t) := e^{tY} \cdot o = \left(\begin{array}{c|c} 0 & \begin{matrix} -e^{2t} & 0 \\ 0 & -1 \end{matrix} \\ \hline \begin{matrix} e^{-2t} & 0 \\ 0 & 1 \end{matrix} & 0 \end{array} \right) \subset S_4$ is contained in the flat

$$\left\{ \begin{pmatrix} 0 & 0 & -\lambda_1 & 0 \\ 0 & (\frac{1}{\lambda_2} - \lambda_2) \sin\theta\cos\theta & 0 & -\frac{1}{\lambda_2}\sin^2\theta - \lambda_2 \cos^2\theta \\ \frac{1}{\lambda_1} & 0 & 0 & 0 \\ 0 & \lambda_2 \sin^2\theta + \frac{1}{\lambda_2}\cos^2\theta & 0 & (\lambda_2 - \frac{1}{\lambda_2})\sin\theta\cos\theta \end{pmatrix} : \lambda_1, \lambda_2 > 0 \right\}.$$

Example 2c. A geodesic c in $\mathbb{H}^2 \times \mathbb{H}^2$ is of the form

$$c(t) = (c_1(t\cos\theta), c_2(t\sin\theta)),$$

where c_i are geodesics in the i-th $\mathbb{H}^2$-factor, $i = 1, 2$, and $\theta \in [0, \pi/2]$. The geodesic c is regular if the parameter θ is contained in the open interval $(0, \pi/2)$, c is singular if $\theta \in \{0, \pi/2\}$. In other words, c is singular if and only if its projection to one of the factors is a point.

2.2 Roots and root spaces

Recall that $\Theta : \mathfrak{g} \to \mathfrak{g}$ is the Cartan involution, and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the Killing form on $\mathfrak{g}$. For this section we will use the following positive definite bilinear form on $\mathfrak{g}$

$$\langle\langle X, Y \rangle\rangle := -B(X, \Theta(Y)), \qquad X, Y \in \mathfrak{g},$$

which is obtained from the Killing form by "changing sign on $\mathfrak{k}$". Notice that on $\mathfrak{p}$ the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ coincides with the one induced by $B|_{\mathfrak{p}}$ (which by our assumption made at the beginning of Section 2 determines the Riemannian structure on S).

Lemma 2.8. *The operator* $\mathrm{ad}\, X$*,* $X \in \mathfrak{p}$*, is self-adjoint on* $\mathfrak{g}$ *with respect to* $\langle\langle \cdot, \cdot \rangle\rangle$*.*

Proof. Let $X \in \mathfrak{p}$. We show that for all $Y, Z \in \mathfrak{g}$

$$\langle\langle (\mathrm{ad}\, X) Y, Z \rangle\rangle = \langle\langle Y, (\mathrm{ad}\, X) Z \rangle\rangle.$$

Indeed, we have $\langle\langle [X,Y], Z\rangle\rangle = -B([X,Y], \Theta Z) = -B(\Theta Z, [X,Y])$; using Proposition 1.19 (1), we conclude

$$\begin{aligned}\langle\langle [X,Y], Z\rangle\rangle &= B(\Theta Z, [Y,X]) \\ &= -B(Y, [\overbrace{-X}^{=\Theta X}, \Theta Z]) = -B(Y, [\Theta X, \Theta Z]) = -B(Y, \Theta[X,Z]) \\ &= \langle\langle Y, [X,Z]\rangle\rangle. \end{aligned}$$ □

Corollary 2.9. *If $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian, then $\{\mathrm{ad}\, H : H \in \mathfrak{a}\}$ is a commutative family of self-adjoint operators on $\mathfrak{g}$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle$.*

This corollary in particular implies that $\mathfrak{g}$ decomposes into an orthogonal direct sum of common eigenspaces with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. This motivates the following

Definition 2.10. A linear map $\alpha\colon \mathfrak{a} \to \mathbb{R}$ is called a *root of the pair* $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \neq \{0\}$. The subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ is then called a *root space*.

It is easy to see that $\mathfrak{a} \subseteq \mathfrak{g}_0$, where the subscript 0 denotes the trivial linear map. We will write Σ for the set of non-trivial roots. We have $\#\Sigma < \infty$, and

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha. \tag{2.2}$$

Recall that $\Theta\colon \mathfrak{g} \to \mathfrak{g}$ denotes the Cartan involution. It is easy to see that for all $X \in \mathfrak{g}$ we have $X + \Theta X \in \mathfrak{k}$, and $X - \Theta X \in \mathfrak{p}$.

Lemma 2.11. *For all $\alpha \in \Sigma$ we have*

$$\Theta\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}.$$

Proof. Let $X \in \mathfrak{g}_\alpha$. Then for all $H \in \mathfrak{a}$ we have $[H,X] = \alpha(H)X$. Moreover, $H \in \mathfrak{p}$, i.e. $\Theta H = -H$. We conclude, using the fact that Θ is a Lie algebra automorphism, $[H, \Theta X] = [\Theta(\underbrace{\Theta H}_{=-H}), \Theta X] = \Theta\big([-H,X]\big) = \Theta\big(-\alpha(H)X\big) = -\alpha(H)\Theta X$. □

Lemma 2.12. *We have*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] := \{[X,Y] : X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta} \quad \textit{for all } \alpha, \beta \in \Sigma.$$

Proof. Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$. Then for $H \in \mathfrak{a}$ we have by the definition of $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and the Jacobi identity

$$\begin{aligned}(\mathrm{ad}\, H)[X,Y] = [H,[X,Y]] &= -[X,[Y,H]] - [Y,[H,X]] \\ &= -[X, -\beta(H)Y] - [Y, \alpha(H)X] \\ &= \beta(H)[X,Y] - \alpha(H)[Y,X] \\ &= (\alpha+\beta)(H)[X,Y]. \end{aligned}$$ □

Lemma 2.13. *$H \in \mathfrak{a} \setminus \{0\}$ is regular if and only if $\alpha(H) \neq 0$ for any $\alpha \in \Sigma$.*

Proof. Let $H \in \mathfrak{a} \setminus \{0\}$. First assume that H is regular, i.e. $Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$ is maximal abelian. Since $\mathfrak{a} \subseteq Z_{\mathfrak{g}}(H)$ this implies $\mathfrak{a} = Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$. Suppose there exists $\alpha \in \Sigma$ such that $\alpha(H) = 0$. Then for all $X \in \mathfrak{g}_\alpha$ we have $[H, X] = \alpha(H)X = 0$, hence $\mathfrak{g}_\alpha \subseteq Z_{\mathfrak{g}}(H)$. Similarly we have $\mathfrak{g}_{-\alpha} \in Z_{\mathfrak{g}}(H)$, and therefore $X - \Theta X \in Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a}$ for all $X \in \mathfrak{g}_\alpha$, a contradiction.

Conversely suppose $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, and $Z_{\mathfrak{g}}(H) \cap \mathfrak{p}$ is not maximal abelian. Then there exists $Y \in \mathfrak{p} \subseteq \mathfrak{g}$, $Y \notin \mathfrak{a}$, $Y \neq 0$ such that $[H, Y] = 0$. For $\alpha \in \Sigma$ denote by Y_α the projection of Y to $\mathfrak{g}_\alpha$. Then

$$0 = [H, Y] = \sum_{\alpha \in \Sigma} [H, Y_\alpha] = \sum_{\alpha \in \Sigma} \alpha(H) Y_\alpha.$$

Since $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$ and $\mathfrak{g}$ is a direct sum of the $\mathfrak{g}_\alpha$, this implies $Y_\alpha = 0$ for all $\alpha \in \Sigma$, a contradiction to $Y \neq 0$. □

Corollary 2.14. *If $\mathfrak{a}_{\mathrm{reg}}$ denotes the set of regular vectors in $\mathfrak{a}$, then*

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker(\alpha).$$

Definition 2.15. A *Weyl chamber* in $\mathfrak{a}$ is a connected component of $\mathfrak{a}_{\mathrm{reg}}$.

Notice that a Weyl chamber is isomorphic to an open Euclidean cone in $\mathfrak{a}$.

In the sequel E_{ij} will denote a quadratic matrix which has a 1 at the position i-th row, j-th column, and zeros everywhere else. The size of the matrix will be taken so that it fits into the frame of the example considered.

Example 1. For $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ we consider $H = \mathrm{Diag}(t_1, t_2, \ldots, t_n) \in \mathfrak{a}$. An easy calculation shows that

$$(\mathrm{ad}\, H) E_{ij} = [H, E_{ij}] = (t_i - t_j) E_{ij}.$$

Hence we have $n(n-1)$ non-zero roots, and $\mathfrak{g}_0 = \mathfrak{a}$. In particular

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{a} + \sum_{i \neq j} \mathbb{R} \cdot E_{ij}.$$

A Weyl chamber in $\mathfrak{a}$ is e.g. given by

$$\mathfrak{a}^+ := \left\{ \mathrm{Diag}(t_1, t_2 \ldots, t_n) : \textstyle\sum_{i=1}^n t_i = 0,\ t_1 > t_2 > \cdots > t_n \right\}.$$

Example 2a. Recall that $M(p, q)$ denotes the set of $(p \times q)$-matrices with values in $\mathbb{R}$. For $\mathrm{SO}(2,3) / \big(\mathrm{SO}(2) \times \mathrm{SO}(3)\big)$ we have

$$\mathfrak{a} = \left\{ H(t_1, t_2) := \left(\begin{array}{cc|ccc} \multicolumn{2}{c|}{0} & t_1 & 0 & 0 \\ & & 0 & t_2 & 0 \\ \hline t_1 & 0 & & & \\ 0 & t_2 & & 0 & \\ 0 & 0 & & & \end{array} \right) : t_1, t_2 \in \mathbb{R} \right\} \cong \mathbb{R}^2.$$

Here we have 8 roots: if $H = H(t_1, t_2)$ then a calculation shows that with α_1, α_2 defined by $\alpha_1(H) = t_1$, $\alpha_2(H) = t_2$, the set of roots is given by

$$\{\alpha_1, \alpha_2, -\alpha_1, -\alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2, -\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}.$$

The corresponding root spaces are

$$\begin{aligned} \mathfrak{g}_{\alpha_1} &= \mathbb{R} \cdot \big(E_{15} + E_{51} + E_{35} - E_{53}\big), & \mathfrak{g}_{-\alpha_1} &= \mathbb{R} \cdot \big(E_{15} + E_{51} - E_{35} + E_{53}\big), \\ \mathfrak{g}_{\alpha_2} &= \mathbb{R} \cdot \big(E_{25} + E_{52} + E_{45} - E_{54}\big), & \mathfrak{g}_{-\alpha_2} &= \mathbb{R} \cdot \big(E_{25} + E_{52} - E_{45} + E_{54}\big), \end{aligned}$$

$$\begin{aligned} \mathfrak{g}_{\alpha_1+\alpha_2} &= \mathbb{R} \cdot \big(E_{12} - E_{21} + E_{23} + E_{32} - E_{14} - E_{41} - E_{34} + E_{43}\big), \\ \mathfrak{g}_{\alpha_1-\alpha_2} &= \mathbb{R} \cdot \big(E_{12} - E_{21} + E_{23} + E_{32} + E_{14} + E_{41} + E_{34} - E_{43}\big), \\ \mathfrak{g}_{-\alpha_1+\alpha_2} &= \mathbb{R} \cdot \big(E_{12} - E_{21} - E_{23} - E_{32} - E_{14} - E_{41} + E_{34} - E_{43}\big), \\ \mathfrak{g}_{-\alpha_1-\alpha_2} &= \mathbb{R} \cdot \big(E_{12} - E_{21} - E_{23} - E_{32} + E_{14} + E_{41} - E_{34} + E_{43}\big). \end{aligned}$$

A Weyl chamber in $\mathfrak{a}$ is for example $\mathfrak{a}^+ = \{H(t_1, t_2) \in \mathfrak{a} : t_1 > t_2 > 0\}$.

Example 2b. For $\mathrm{Sp}(4, \mathbb{R})/\big(\mathrm{SO}(4) \cap \mathrm{Sp}(4, \mathbb{R})\big)$ we have

$$\mathfrak{a} = \{H(t_1, t_2) := \mathrm{Diag}(t_1, t_2, -t_1, -t_2) : t_1, t_2 \in \mathbb{R}\} \cong \mathbb{R}^2.$$

If $H = H(t_1, t_2)$ then with α_1, α_2 defined by $\alpha_1(H) = t_1$, $\alpha_2(H) = t_2$, the set of roots is given by

$$\{2\alpha_1, 2\alpha_2, -2\alpha_1, -2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2, -\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}.$$

Here the corresponding root spaces are

$$\begin{gathered} \mathfrak{g}_{2\alpha_1} = \mathbb{R} \cdot E_{13}, \quad \mathfrak{g}_{2\alpha_2} = \mathbb{R} \cdot E_{24}, \quad \mathfrak{g}_{-2\alpha_1} = \mathbb{R} \cdot E_{31}, \quad \mathfrak{g}_{-2\alpha_2} = \mathbb{R} \cdot E_{42}, \\ \mathfrak{g}_{\alpha_1+\alpha_2} = \mathbb{R} \cdot \big(E_{14} + E_{23}\big), \quad \mathfrak{g}_{\alpha_1-\alpha_2} = \mathbb{R} \cdot \big(E_{12} - E_{43}\big), \\ \mathfrak{g}_{-\alpha_1+\alpha_2} = \mathbb{R} \cdot \big(E_{21} - E_{34}\big), \quad \mathfrak{g}_{-\alpha_1-\alpha_2} = \mathbb{R} \cdot \big(E_{32} + E_{41}\big), \end{gathered}$$

and a possible Weyl chamber in $\mathfrak{a}$ is $\mathfrak{a}^+ = \{H(t_1, t_2) \in \mathfrak{a} : t_1 > t_2 > 0\}$.

Notice that even though at first sight this root system looks different from the one in Example 2a, the root systems are isomorphic: taking instead of α_1, α_2 the roots μ_1, μ_2 defined by $\mu_1(H) = t_1 + t_2$ and $\mu_2(H) = t_1 - t_2$, the set of roots equals

$$\{\mu_1 + \mu_2, \mu_1 - \mu_2, -\mu_1 - \mu_2, -\mu_1 + \mu_2, \mu_1, \mu_2, -\mu_2, -\mu_1\},$$

and the corresponding root spaces obviously are

$$\begin{gathered} \mathfrak{g}_{\pm\mu_1} = \mathfrak{g}_{\pm(\alpha_1+\alpha_2)}, \quad \mathfrak{g}_{\pm\mu_2} = \mathfrak{g}_{\pm(\alpha_1-\alpha_2)}, \\ \mathfrak{g}_{\pm(\mu_1+\mu_2)} = \mathfrak{g}_{\pm 2\alpha_1}, \quad \mathfrak{g}_{\pm(\mu_1-\mu_2)} = \mathfrak{g}_{\pm 2\alpha_2}. \end{gathered}$$

The fact that the two root systems are the same has a deeper reason: the Lie algebras $\mathfrak{so}(2,3)$ and $\mathfrak{sp}(4, \mathbb{R})$ are isomorphic (see e.g. Chapter X, §6.4 (ii) in [H]).

Remark 2.16. The roots of the pair $(\mathfrak{g}, \mathfrak{a})$ form a root system in the finite dimensional vector space $\mathfrak{a}$ over $\mathbb{R}$ according to the following definition.

Definition 2.17 ([H], X.3.1). Let V be a finite dimensional vector space over $\mathbb{R}$ and $R \subset V$ a finite set of non-zero vectors. R is called a *root system* in V, and its members are called *roots* if

(1) R generates V;

(2) for each $\alpha \in R$ there exists a reflection s_α along α leaving R invariant;

(3) for all $\alpha, \beta \in R$ the number $m_{\alpha\beta}$ determined by

$$s_\alpha \beta = \beta - m_{\alpha\beta}\alpha$$

is an integer, i.e. $m_{\alpha\beta} \in \mathbb{Z}$.

2.3 Iwasawa decomposition

For this section we fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and denote by $r := \dim \mathfrak{a}$ the rank of S. We will need the following subset of the set of roots Σ of the pair $(\mathfrak{g}, \mathfrak{a})$

$$\Sigma^+ := \{\alpha \in \Sigma : \alpha(H) > 0 \text{ for all } H \in \mathfrak{a}^+\}.$$

Definition 2.18. A root is called *simple* if it cannot be written as a sum $\alpha = \beta + \gamma$, where $\beta, \gamma \in \Sigma^+$.

By Theorem III.V.7 in [H] there exist a set of r simple roots $\Upsilon := \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $c_1, \ldots, c_r \in \mathbb{N} \cup \{0\}$ such that

$$\alpha = \sum_{i=1}^{r} c_i \alpha_i \quad \text{for all } \alpha \in \Sigma^+.$$

Such a set is called a *fundamental set of roots*. Moreover, we have

$$\Sigma = \Sigma^+ \sqcup (-\Sigma^+).$$

We next put

$$\mathfrak{n}^+ := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha,$$

which is a nilpotent Lie algebra by Lemma 2.12 and the fact that Σ is a finite set. Set $N^+ := e^{\mathfrak{n}^+}$ which is a unipotent subgroup of G. The following theorem is called the Iwasawa decomposition:

Theorem 2.19 ([H],Theorem IX.1.3). *We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. If $A = e^{\mathfrak{a}}$, then the mapping $K \times A \times N^+ \to G$, $(k, a, n) \mapsto kan$ is a diffeomorphism.*

As a consequence we have $S = G \cdot o = N^+ A \cdot o$ which is sometimes called the "foliation by flats". N^+-orbits in S are called *horocycles*.

Example 1. We consider the set Σ^+ of positive roots with respect to the Weyl chamber

$$\mathfrak{a}^+ = \left\{ \mathrm{Diag}(t_1, t_2, \dots, t_n) : \textstyle\sum_{i=1}^n t_i = 0,\ t_1 > t_2 > \dots > t_n \right\}.$$

Let α_i denote the simple root determined by $\alpha_i(\mathrm{Diag}(t_1, t_2, \dots t_n)) = t_i - t_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$. Then a fundamental set of roots is precisely the set $\Upsilon = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$.

The nilpotent Lie algebra $\mathfrak{n}^+$ is the set of upper triangular $(n \times n)$-matrices with zeros in the diagonal, and N^+ is the group of upper triangular $(n \times n)$-matrices with 1's in the diagonal. If $\mathrm{Diag}_+(n)$ denotes the set of diagonal $(n \times n)$-matrices with positive entries and determinant one, we have

$$\mathrm{SL}(n, \mathbb{R}) = \mathrm{SO}(n) \cdot \mathrm{Diag}_+(n) \cdot N^+. \tag{2.3}$$

This decomposition simply comes from the Gram–Schmidt orthonormalization procedure in linear algebra.

In $\mathbb{H}^2$ the foliation by flats is simply the foliation by geodesics given by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot i = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot e^t i = e^t i + x, \quad x, t \in \mathbb{R}.$$

Any fixed $t \in \mathbb{R}$ determines a horocycle $\{e^t i + x : x \in \mathbb{R}\}$.

Example 2b. In $\mathrm{Sp}(4, \mathbb{R}) / \big(\mathrm{SO}(4) \cap \mathrm{Sp}(4, \mathbb{R})\big)$ we take the Weyl chamber

$$\mathfrak{a}^+ = \left\{ \mathrm{Diag}(t_1, t_2, -t_1, -t_2) : t_1 > t_2 > 0 \right\}.$$

Then the set of positive roots in the notation from the previous section is given by $\Sigma^+ = \{2\alpha_1, 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2\}$. The roots $2\alpha_1$ and $\alpha_1 + \alpha_2$ are not simple; a fundamental set of roots is $\Upsilon = \{\alpha_1 - \alpha_2, 2\alpha_2\}$. Here we have

$$\mathfrak{n}^+ = \left\{ \left(\begin{array}{cc|cc} 0 & a & x & z \\ 0 & 0 & z & y \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & -a & 0 \end{array} \right) : a, x, y, z \in \mathbb{R} \right\}.$$

2.4 The space of maximal flats

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. In Section 2.1 we have seen that every set of the form $ge^{\mathfrak{a}} \cdot o$, $g \in G$, is a maximal flat in S. The following theorem shows that every maximal flat can be written in this way.

Theorem 2.20. *All maximal flats are conjugate in S, i.e. the space of maximal flats is homogeneous.*

Proof. We have to show that for arbitrary maximal flats F, F' in S there exists $g \in G$ such that $F' = g \cdot F$. Since G acts transitively on S we may assume that $o \in F$. Pick $x \in F'$ and let $g \in G$ be such that $g \cdot x = o$. Replacing F' by $g \cdot F'$ we may further assume that $o \in F'$. Now let $\mathfrak{a}$, $\mathfrak{a}' \subseteq \mathfrak{p}$ be maximal abelian subspaces such that $F = e^{\mathfrak{a}} \cdot o$ and $F' = e^{\mathfrak{a}'} \cdot o$. We show that there exists $k \in K$ such that $F' = k \cdot F$, i.e. $e^{\mathfrak{a}'} \cdot o = k e^{\mathfrak{a}} \cdot o$. This is equivalent to the existence of $k \in K$ such that $\mathfrak{a}' = \mathrm{Ad}(k)\mathfrak{a}$, hence the claim follows from Theorem 2.6. □

Example 1. Recall from (2.1) that in $\mathrm{Pos}_1(n)$ the set

$$F = \left\{ \mathrm{Diag}(\lambda_1, \ldots, \lambda_n) : \lambda_1, \ldots, \lambda_n > 0,\ \prod_{i=1}^n \lambda_i = 1 \right\}$$

is a maximal flat containing the base point $o = I_n$. Given an arbitrary point $p \in \mathrm{Pos}_1(n)$, how does a flat through o containing p look like?

Since $p \in \mathrm{Pos}_1(n)$ is diagonalizable, there exists $k \in \mathrm{SO}(n)$ such that the matrix kpk^{-1} is diagonal with positive entries and determinant 1. With the action of $\mathrm{SL}(n, \mathbb{R})$ on $\mathrm{Pos}_1(n)$ given by $g \cdot p := g^t pg$, $p \in \mathrm{Pos}_1(n)$, $g \in \mathrm{SL}(n, \mathbb{R})$, we conclude

$$kpk^{-1} = (k^{-1})^t pk^{-1} = k^{-1} \cdot p \in F,$$

hence $F' := k \cdot F$ is a flat in $\mathrm{Pos}_1(n)$ through o containing p.

Notice that the matrix $k \in \mathrm{SO}(n)$ above is not unique. Conjugating with an appropriate element $w \in \mathrm{SO}(n)$ we can arrange that $wkpk^{-1}w^{-1}$ is a diagonal matrix $\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. This motivates the definition of the Weyl group in the following section.

Example 2c. If F is a flat in $\mathbb{H}^2 \times \mathbb{H}^2$, there exist unit speed geodesics c_1, c_2 in the two factors such that $F = \{(c_1(t_1), c_2(t_2)) : t_1, t_2 \in \mathbb{R}\}$. Since $\mathrm{SL}(2, \mathbb{R})$ acts simply transitively on the set of unit speed geodesics of $\mathbb{H}^2$, there exists $(g_1, g_2) \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ such that $g_1 \cdot c_1(t_1) = e^{t_1} i$ and $g_2 \cdot c_2(t_2) = e^{t_2} i$ for all $t_1, t_2 \in \mathbb{R}$.

2.5 Weyl group and opposition involution

We have seen that a maximal flat $F \subset S$ is an isometric copy of $\mathbb{R}^r$, where r denotes the rank of S. Hence abstractly its full isometry group would be $\mathrm{O}(n) \ltimes \mathbb{R}^n$. However, the induced isometries of F (i.e. the isometries of F in $G = \mathrm{Is}^o(S)$) are generated by all translations, but only *finitely many* rotations. These rotations are encoded in the so-called Weyl group of S.

We denote by M the centralizer, and by M^* the normalizer of $\mathfrak{a}$ in K, i.e.

$$M = \{k \in K : \mathrm{Ad}(k)H = H \text{ for all } H \in \mathfrak{a}\},$$
$$M^* = \{k \in K : \mathrm{Ad}(k)H \in \mathfrak{a} \text{ for all } H \in \mathfrak{a}\}.$$

Definition 2.21. The *Weyl group* of S is the factor group M^*/M.

The Weyl group is finite and satisfies the following properties which are proved in Chapter VII of [H].

Proposition 2.22 ([H], VII.2). (1) *W leaves invariant the configuration of singular hyperplanes.*

(2) *W is finitely generated by reflections s_α (in the walls of a fixed Weyl chamber).*
(3) *W acts simply transitively on the set of Weyl chambers of a flat with apex o.*

Remark 2.23. If S is a rank one symmetric space, then the Weyl group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Given a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, there exists a unique element $w_* \in W$ such that every representative $m_{w_*} \in M^*$ of w_* satisfies

$$\mathrm{Ad}(m_{w_*})\mathfrak{a}^+ = -\mathfrak{a}^+ := \{-H \in \mathfrak{a} : H \in \mathfrak{a}^+\}.$$

If $\overline{\mathfrak{a}^+}$ denotes the closure of the Weyl chamber $\mathfrak{a}^+$, then this element defines a map

$$\begin{aligned} \iota\colon \overline{\mathfrak{a}^+} &\to \overline{\mathfrak{a}^+} \\ H &\mapsto -\mathrm{Ad}(m_{w_*})H, \end{aligned} \tag{2.4}$$

which is called the *opposition involution*. Notice that ι is an isometry which is the identity if and only if $\mathrm{Ad}(m_{w_*}) = -\mathrm{id}_{\mathfrak{a}}$.

Example 1. For $\mathrm{Pos}_1(n)$ the Weyl group is isomorphic to the group of permutations of n-tuples. The element w_* corresponds to the permutation which maps the n-tuple $(t_1, t_2, \dots, t_n)$ to $(t_n, t_{n-1}, \dots, t_1)$. For the closed Weyl chamber

$$\overline{\mathfrak{a}^+} = \left\{\mathrm{Diag}(t_1, t_2, \dots, t_n) : \textstyle\sum_{i=1}^n t_i = 0,\ t_1 \geq t_2 \geq \cdots \geq t_n\right\} \subset \mathrm{sym}_0(n)$$

the opposition involution is given by

$$\iota\big(\mathrm{Diag}(t_1, t_2, \dots, t_n)\big) = \mathrm{Diag}(-t_n, -t_{n-1}, \dots, -t_1).$$

Example 2b. For S_{2q} the Weyl group is isomorphic to the subgroup of permutations of $2q$-tuples generated by transpositions among the first q elements and the transpositions $(k, k+q)$, $1 \leq k \leq q$. The element w_* here acts as $-\mathrm{id}_{\mathfrak{a}}$, so $\iota = \mathrm{id}_{\overline{\mathfrak{a}^+}}$.

Example 2c. For $\mathbb{H}^2 \times \mathbb{H}^2$ the Weyl group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The opposition involution is again the identity.

Definition 2.24. By abuse of notation a *Weyl chamber* in S is defined to be a set of the form $ge^{\mathfrak{a}^+} \cdot o$, where $\mathfrak{a}^+$ is a Weyl chamber in $\mathfrak{a}$ and $g \in G$.

2.6 Cartan decomposition and Cartan vector

This section provides a refinement of the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G studied in Section 1.3. We fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ in a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\overline{\mathfrak{a}^+}$ its closure. The following theorem implies that symmetric spaces of non-compact type are Weyl chamber isotropic.

Theorem 2.25.
$$\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)\overline{\mathfrak{a}^+}.$$

Proof. Let $X \in \mathfrak{p}$ be arbitrary. By Theorem 2.6 there exists $k \in K$ such that $\mathrm{Ad}(k)X \in \mathfrak{a}$. The claim now follows from the fact that the Weyl group acts transitively on the set of Weyl chambers of $\mathfrak{a}$. □

Since S is complete, the Theorem of Hopf–Rinow implies that the base point $o \in S$ can be joined to any point $x \in S$ by a geodesic. Moreover, since $D_e\tau : \mathfrak{p} \to T_oS$ is an isomorphism and $\exp_o\big(D_e\tau(X)\big) = e^X \cdot o$, we have $S = e^{\mathfrak{p}} \cdot o$. So S is diffeomorphic to $\mathbb{R}^{\dim \mathfrak{p}}$ and we obtain from the previous theorem *polar coordinates* for S:

Corollary 2.26. $S = e^{\mathfrak{p}} \cdot o = Ke^{\overline{\mathfrak{a}^+}} \cdot o.$

If $x = ke^H \cdot o \in S$, we will call $k \in K$ an *angular projection* and $H \in \overline{\mathfrak{a}^+}$ the *Cartan projection* of x. It can be shown (see e.g. [H], Theorem IX.1.1) that the Cartan projection of a point x is unique, whereas its angular projection in general is not. Using the fact that $S \cong G/K$, we get the *Cartan decomposition* of G:

Corollary 2.27. $G = Ke^{\overline{\mathfrak{a}^+}}K.$

The following definition plays an important role in the theory of higher rank symmetric spaces:

Definition 2.28. Given $x, y \in S$, we choose $g \in G$ such that $g \cdot x = o$. The *Cartan vector* $H(x, y) \in \overline{\mathfrak{a}^+}$ of the ordered pair of points $(x, y) \in S \times S$ is defined as the Cartan projection of $g \cdot y$.

Notice that the definition of the Cartan vector $H(x, y)$ does not depend on the choice of $g \in G$ such that $g \cdot x = o$. Indeed, if $h \in G$ also satisfies $h \cdot x = o$, then $hg^{-1} \in K$. So if $g \cdot y = ke^H \cdot o$ we get
$$h \cdot y = hg^{-1}g \cdot y = \underbrace{hg^{-1}k}_{\in K}\, e^H \cdot o.$$
Furthermore, the length of the Cartan vector of the ordered pair of points $(x, y) \in S\times S$ is simply the distance $d(x, y)$. In particular, if S is a rank one symmetric space, then the Cartan vector reduces to the Riemannian distance. Hence in higher rank symmetric spaces the Cartan vector is a natural generalization of the Riemannian distance function.

We remark that $H(y, x) = \iota(H(x, y))$, where ι is the opposition involution (2.4). This can be seen as follows: let $h \in G$ be such that $h \cdot x = o$ and $h \cdot y = e^{H(x,y)} \cdot o$, and $m_{w_*} \in M^*$ be a representative of $w_* \in W$. Then $g := e^{-H(x,y)}h \in G$ satisfies

$g \cdot y = o$, and we have

$$\begin{aligned} g \cdot x &= e^{-H(x,y)} h \cdot x = e^{-H(x,y)} \cdot o \\ &= (m_{w_*})^{-1} e^{\iota(H(x,y))} m_{w_*} \cdot o = (m_{w_*})^{-1} e^{\iota(H(x,y))} \cdot o, \end{aligned}$$

hence the Cartan projection $H(y,x)$ of $g \cdot x$ equals $\iota(H(x,y))$.

3 The geometry at infinity

In this section we will describe the geometry at infinity of a globally symmetric space S of non-compact type. From Section 1.1 and Theorem 1.29 we know that S is in particular a Hadamard manifold, i.e. a complete simply connected Riemannian manifold of non-positive sectional curvature. Therefore S is homeomorphic to $\mathbb{R}^{\dim S}$ and can be compactified by attaching its so-called geometric boundary. Due to the rich algebraic structure of globally symmetric spaces, this boundary can be described much more precisely than it is possible for general Hadamard manifolds. In particular, there exists a natural quotient of a dense subset of the geometric boundary which is called the Furstenberg boundary; for rank one symmetric spaces these two boundaries coincide.

On the other hand, we are led to study the pairs of points in the geometric boundary which can be joined by a geodesic. For rank one symmetric spaces all pairs of distinct boundary points can be joined by a geodesic; in the higher rank setting the flats destroy this property. However, the Bruhat decomposition will allow us to describe the pairs of boundary points which can be joined by a geodesic.

Finally we will introduce Busemann functions which serve as a tool in the construction of G-invariant Finsler pseudo-distances on S. When studying the action of discrete groups on S, these pseudo-distances play a key role for the construction of generalized Patterson–Sullivan measures (see e.g. [A], [L2]). However, we will not touch on this subject here.

Recall the notation from Section 2: $o \in S$ denotes the base point, $G = \mathrm{Is}^o(S)$ and $K \subset G$ the compact isotropy subgroup at o. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition and assume that the Riemannian structure of S is induced by the Killing form restricted to $\mathfrak{p}$. Moreover, we will fix once and for all a Weyl chamber $\mathfrak{a}^+$ in a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Throughout this section all geodesics and geodesic rays are supposed to have unit speed.

3.1 The geometric boundary of S

Definition 3.1. We say that two geodesic rays c_1, c_2 are *equivalent* if

$$d(c_1(t), c_2(t)) \quad \text{is bounded as } t \to \infty.$$

The *geometric boundary* ∂S of S is defined as the set of geodesic rays in S modulo this equivalence relation.

In order to topologize the space $\overline{S} := S \cup \partial S$, we introduce the following sets: for $\varepsilon > 0$, $R \gg 1$, $x \in S$ and $\eta \in \partial S$ let $\mathrm{C}_{x,\eta}^{R,\varepsilon} \subset \overline{S}$ be the *truncated cone*

$$\mathrm{C}_{x,\eta}^{R,\varepsilon} := \{y \in \overline{S} : d(x,y) > R,\ d(c_{x,\eta}(R), c_{x,y}(R)) < \varepsilon\}$$

in $\overline{S}$, where $c_{x,\eta}$ denotes the unique unit speed geodesic emanating from $x \in S$ in the class of $\eta \in \partial S$ (compare (1.1) for the definition of $c_{x,y}$, $y \in S$).

Definition 3.2 ([Ba], Chapter II). The *cone topology* on $\overline{S}$ is the topology generated by the open sets in S and these truncated cones.

If not stated otherwise, convergence in $S \cup \partial S$ will always mean convergence with respect to the cone topology. The relative topology on ∂S turns the geometric boundary into a topological space.

The isometry group of S has a natural action by homeomorphisms on the geometric boundary. If $g \in G$, and $\xi \in \partial S$ is represented by a geodesic ray c in S, then $g \cdot \xi$ is the class of the geodesic ray $g \cdot c$ in S. Notice that this assignment does not depend on the choice of the geodesic ray c in the class of ξ: indeed, if c' is a ray different from c representing ξ, then $d(c(t), c'(t))$ is bounded as t tends to infinity. Since g is an isometry, $d(g \cdot c(t), g \cdot c'(t))$ is bounded as t tends to ∞, hence $g \cdot c'$ is equivalent to $g \cdot c$ and therefore represents the same point in the geometric boundary.

It is well-known that the geometric boundary endowed with the cone topology is homeomorphic to the unit tangent space of an arbitrary point $x \in S$. If $\mathfrak{p}_1$ and $\overline{\mathfrak{a}_1^+}$ denote the set of vectors of length 1 with respect to the Killing form in $\mathfrak{p}$ and $\overline{\mathfrak{a}^+}$ respectively, then

$$\partial S \cong T_o^1 S \cong \mathfrak{p}_1 = \mathrm{Ad}(K)\overline{\mathfrak{a}_1^+}.$$

In particular, a tuple $(k, H) \in K \times \overline{\mathfrak{a}_1^+}$ defines a unique point in ∂S by taking the class of the geodesic ray $c(t) := ke^{Ht} \cdot o$, $t > 0$.

Conversely, given a point $\xi \in \partial S$ there exists $k \in K$ and $H \in \overline{\mathfrak{a}_1^+}$ such that ξ is the class of the geodesic ray $c(t) := ke^{Ht} \cdot o$, $t > 0$. In this case we write $\xi = c(\infty)$. By the Cartan decomposition, H is uniquely determined by ξ, whereas k is only determined up to right multiplication by an element in the centralizer of H in K. We call k an *angular projection*, and H the *Cartan projection* of ξ, and we will write $\xi = (k, H)$.

If $r = \mathrm{rank}(S) > 1$, we define the *regular boundary* $\partial S^{\mathrm{reg}}$ as the set of classes with Cartan projection in $\mathfrak{a}_1^+$. If $\mathrm{rank}(S) = 1$, we use the convention $\partial S^{\mathrm{reg}} = \partial S$.

Notice that $G \cdot \xi = K \cdot \xi$ for any $\xi \in \partial S$. Furthermore, G acts transitively on ∂S if and only if $\mathrm{rank}(S) = 1$.

Example 1. For $n = 2$, the symmetric space $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ can be identified with $\mathbb{H}^2$. The geometric boundary of the hyperbolic plane $\mathbb{H}^2$ is the set $\mathbb{R} \cup \{\infty\}$ which is homeomorphic to the sphere $\mathbb{S}^1$.

In general, for $n \geq 2$, a point ξ in the geometric boundary of $S = \mathrm{Pos}_1(n)$ determines an eigenvalue-flag pair as follows: let $X = X(\xi) \in \mathfrak{p} \cong T_oS$ be the unit vector such that the geodesic ray $c(t) := e^{Xt} \cdot o$, $t > 0$, satisfies $c(\infty) = \xi$. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be the $l \leq n$ distinct eigenvalues of X, arranged so that $\lambda_1 > \lambda_2 > \cdots > \lambda_l$. For $1 \leq i \leq l$ let E_i be the eigenspace of X in $\mathbb{R}^n$ for the eigenvalue λ_i, $m_i := \dim E_i$, and V_i the direct sum of the eigenspaces $\{E_j : 1 \leq j \leq i\}$. We thus obtain a flag of subspaces $V_1 \subset V_2 \subset \cdots \subset V_l = \mathbb{R}^n$. Notice that $X \in \mathfrak{p} = \mathrm{sym}_0(n)$ implies $\mathrm{Tr}(X) = 0$, hence $\sum_{i=1}^{l} m_i\lambda_i = 0$. Moreover, $B(X, X) = 1$ translates into the condition $\sum_{i=1}^{l} m_i\lambda_i^2 = 1$. Hence to each point $\xi \in \partial S$ we have associated a vector $\lambda(\xi) = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{R}^l$ and a flag $F(\xi) = (V_1, V_2, \ldots, V_l)$ in $\mathbb{R}^n$ subject to the above conditions. Such a pair will be called an eigenvalue-flag pair.

The group $G = \mathrm{SL}(n, \mathbb{R})$ acts naturally on the flags in $\mathbb{R}^n$: if $g \in G$, then a subspace $V \subseteq \mathbb{R}^n$ is mapped by g to the subspace $g \cdot V \subseteq \mathbb{R}^n$ of the same dimension. So a flag $F = (V_1, V_2, \ldots, V_l)$ is mapped to the flag $g \cdot F := (g \cdot V_1, g \cdot V_2, \ldots, g \cdot V_l)$. We can therefore consider the action of $G = \mathrm{SL}(n, \mathbb{R})$ on the set of eigenvalue-flag pairs given by

$$g \cdot (\lambda, F) := (\lambda, g \cdot F), \quad g \in G.$$

For $1 \leq j \leq n$ we denote by $e(j)$ the j-th standard basis vector in $\mathbb{R}^n$. Suppose

$$X(\xi) \in \overline{\mathfrak{a}^+} = \left\{\mathrm{Diag}(t_1, t_2, \ldots, t_n) : t_1 \geq \cdots \geq t_n, \textstyle\sum_{i=1}^{n} t_i = 0\right\}.$$

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$, $2 \leq l \leq n$, be the distinct eigenvalues of $X(\xi)$ in decreasing order, and m_i, $1 \leq i \leq l$, the multiplicity of the eigenvalue λ_i. Then with $d_i := \sum_{j=1}^{i} m_j$ the flag $F(\xi) = (U_1, U_2, \ldots, U_l)$ is given by

$$U_i = \mathrm{span}_{\mathbb{R}}(e(1), \ldots, e(d_i)), \quad 1 \leq i \leq l. \tag{3.1}$$

This flag will be called the *standard flag in* $\mathbb{R}^n$ *determined by* $(m_1, m_2, \ldots, m_l)$.

Conversely, given an integer l with $2 \leq l \leq n$, a flag of subspaces $F = (V_1, V_2, \ldots, V_l)$ in $\mathbb{R}^n$, $m_i := \dim V_i - \dim V_{i-1}$, $1 \leq i \leq l$, and a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{R}^l$ satisfying

$$\lambda_1 > \lambda_2 > \cdots > \lambda_l, \quad \sum_{i=1}^{l} m_i\lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{l} m_i\lambda_i^2 = 1,$$

then there exists a unique point $\xi \in \partial S$ such that $\lambda(\xi) = \lambda$ and $F(\xi) = F$ as follows. Let $H \in \mathrm{sym}_0(n)$ be the diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_l$ occurring according to their multiplicities. We then choose an element $k \in K = \mathrm{SO}(n)$ such that $k \cdot F$ is the standard flag in $\mathbb{R}^n$ determined by $(m_1, m_2, \ldots, m_l)$; this is possible because we can choose an orthonormal basis for V_1, and if $i \geq 2$ we can extend the orthonormal basis of V_{i-1} to an orthonormal basis of V_i. Moreover, different choices of such $k \in K$ are equal up to left multiplication by an element in K which preserves the standard flag in $\mathbb{R}^n$ determined by $(m_1, m_2, \ldots, m_l)$; hence all possible choices of $k \in K$ define the same ray $c(t) := k^{-1}e^{Ht} \cdot o$, $t > 0$, and we can set $\xi = c(\infty)$.

So we have seen that the geometric boundary ∂S of $S = \mathrm{Pos}_1(n)$ is identified with the set of eigenvalue-flag pairs. Notice that $\xi \in \partial S^{\mathrm{reg}}$ if and only if $l = n$.

Moreover, this identification is G-equivariant: it can be shown that for $g \in G$ and $\xi \in \partial S$ with corresponding eigenvalue-flag pair $(\lambda(\xi), F(\xi))$ we have

$$\big(\lambda(g \cdot \xi), F(g \cdot \xi)\big) = g \cdot \big(\lambda(\xi), F(\xi)\big).$$

Example 2b. Consider the space $S = S_{2q}$ of ω-compatible complex structures on $(\mathbb{R}^{2q}, \omega)$ with base point $o \in S_{2q}$ given by the matrix (1.6). A point ξ in the geometric boundary ∂S_{2q} is uniquely determined by an element $X = X(\xi) \in \mathfrak{p}_1$ such that $c_{o,\xi}(t) = e^{Xt} \cdot o$. Now $X \in \mathrm{sym}_0(2q) \cap \mathfrak{sp}(2q, \mathbb{R})$ implies that X possesses q pairs of eigenvalues $(\lambda, -\lambda)$ with $\lambda \geq 0$. Denote by $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ the $l \leq q$ distinct positive eigenvalues of X, arranged so that $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. For $1 \leq i \leq l$ let $E_i \subset \mathbb{R}^{2q}$ be the eigenspace of X for the eigenvalue λ_i, $m_i := \dim E_i$, and W_i the direct sum of the eigenspaces $\{E_j : 1 \leq j \leq i\}$. Notice that the subspaces W_i, $1 \leq i \leq l$, are isotropic, i.e.

$$W_i \subseteq W_i^{\omega} := \{x \in \mathbb{R}^{2q} : \omega(x, y) = 0 \text{ for all } y \in W_i\}.$$

We so obtain a flag of isotropic subspaces $W_1 \subset W_2 \subset \ldots \subset W_l$ of the symplectic vector space $(\mathbb{R}^{2q}, \omega)$. Since $B(X, X) = 1$ we further have the condition $2\sum_{i=1}^{k} m_i \lambda_i^2 = 1$. Hence to each point $\xi \in \partial S_{2q}$ we have associated a vector $\lambda(\xi) = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{R}^l$ and a flag $F(\xi) = (W_1, W_2, \ldots, W_l)$ of isotropic subspaces in $\mathbb{R}^{2q}$.

The symplectic group $\mathrm{Sp}(2q, \mathbb{R})$ maps an isotropic subspace of $(\mathbb{R}^{2q}, \omega)$ to an isotropic subspace of $(\mathbb{R}^{2q}, \omega)$. So $G = \mathrm{Sp}(2q, \mathbb{R})$ acts naturally on the set of isotropic flags as above: if $g \in G$ and $F = (W_1, W_2, \ldots, W_l)$ is an isotropic flag, then $g \cdot F$ is the isotropic flag $g \cdot F := (g \cdot W_1, g \cdot W_2, \ldots, g \cdot W_l)$. As in the previous example we will consider the action of $G = \mathrm{Sp}(2q, \mathbb{R})$ on the set of pairs of positive eigenvalues and isotropic flags given by

$$g \cdot (\lambda, F) := (\lambda, g \cdot F), \quad g \in G.$$

Let $e(j)$ denote the j-th standard basis vector in $\mathbb{R}^{2q}$, $1 \leq j \leq 2q$, and assume that

$$X(\xi) \in \overline{\mathfrak{a}^+} = \left\{ \left(\begin{smallmatrix} D & 0 \\ 0 & -D \end{smallmatrix}\right) : D = \mathrm{Diag}(t_1, t_2, \ldots, t_q),\ t_1 \geq t_2 \geq \cdots \geq t_q \right\}.$$

If $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$, $1 \leq l \leq q$, are the distinct positive eigenvalues of $X(\xi)$ in decreasing order, m_i denotes the multiplicity of the eigenvalue λ_i, and $d_i := \sum_{j=1}^{i} m_j$, $1 \leq i \leq l$, then the flag $F(\xi) = (U_1, U_2, \ldots, U_l)$ is given by

$$U_i = \mathrm{span}_{\mathbb{R}}(e(1), \ldots, e(d_i)), \quad 1 \leq i \leq l. \tag{3.2}$$

This flag will be called the *isotropic standard flag in* $(\mathbb{R}^{2q}, \omega)$ *determined by* $(m_1, m_2, \ldots, m_l)$.

Notice that for $1 \le i \le l$ the eigenspace E_i for the eigenvalue λ_i is given by $E_i = \mathrm{span}_{\mathbb{R}}(e(d_{i-1}+1), \dots, e(d_i))$, where we used the convention $d_0 = 0$; moreover, the eigenspace E_{-i} for the eigenvalue $-\lambda_i$ is $E_{-i} = \mathrm{span}_{\mathbb{R}}(e(d_{i-1}+q+1), \dots, e(d_i+q))$. If $d_l < q$, then the eigenspace E_0 for the eigenvalue 0 is the symplectic subspace $E_0 = \mathrm{span}_{\mathbb{R}}(e(d_l+1), \dots, e(q), e(d_l+q+1), \dots, e(2q))$.

Conversely, given an integer l with $1 \le l \le q$, a flag of isotropic subspaces $F = (W_1, W_2, \dots, W_l)$ in $\mathbb{R}^{2q}$, $m_i := \dim W_i - \dim W_{i-1}$, $1 \le i \le l$, and a vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathbb{R}^l$ such that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0 \quad \text{and} \quad 2 \cdot \sum_{i=1}^{l} m_i \lambda_i^2 = 1,$$

then there exists a unique point $\xi \in \partial S$ such that $\lambda(\xi) = \lambda$ and $F(\xi) = F$ as follows: let $D \in M(q,q)$ be the diagonal matrix with entries $\lambda_1, \lambda_2, \dots, \lambda_l$ occurring according to their multiplicities; if $d_l := \sum_{i=1}^{l} m_i < q$, the remaining $q - d_l$ diagonal entries are filled with zeros. Set $H := \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$. Consider the standard scalar product q_0 in $\mathbb{R}^{2q}$ which – as we have seen in Section 1.5 – is given by

$$q_0(x,y) := \omega(x, J_0 y), \quad x, y \in \mathbb{R}^{2q},$$

where J_0 denotes the complex structure given by the matrix (1.6).

Notice that if $d_l = q$, then the subspace $L := W_l$ is necessarily Lagrangian, i.e. $L = L^{\omega}$. If $d_l < q$ we choose a Lagrangian subspace L of $\mathbb{R}^{2q}$ such that $W_l \subset L$. We take an orthonormal basis $\{b_1, \dots, b_{m_1}\}$ of W_1 with respect to q_0 and extend it inductively to an orthonormal basis of $W_2, \dots, W_l, L$. By a standard procedure in symplectic linear algebra (see e.g. Section 1.1 in [dS]) we can inductively extend the orthonormal basis $\{b_1, b_2, \dots, b_q\}$ of L to an orthonormal basis $\{b_1, \dots, b_{2q}\}$ of $\mathbb{R}^{2q}$ such that $\omega(b_i, b_j) = \delta_{i,j-q}$ for all $i, j \in \{1, 2, \dots, 2q\}$. Hence one can find $k \in K = \mathrm{SO}(2q) \cap \mathrm{Sp}(2q, \mathbb{R})$ such that $k \cdot b_i = e(i)$ for all $1 \le i \le 2q$, which implies that $k \cdot F$ is the isotropic standard flag determined by $(m_1, m_2, \dots, m_l)$. Notice that if $k' \in K$, $k' \ne k$, maps F to the isotropic standard flag determined by $(m_1, m_2, \dots, m_l)$, then $k^{-1}k' \in K$ has to leave invariant the eigenspaces of H. This precisely translates to the fact that $\mathrm{Ad}(k^{-1}k')H = H$, hence any such k determines the same geodesic ray $c(t) := k^{-1}e^{Ht} \cdot o$, $t > 0$, and we set $\xi = c(\infty)$.

So we have seen that ∂S can be identified with the set of pairs of positive eigenvalues and isotropic flags. Notice that $\xi \in \partial S^{\mathrm{reg}}$ if and only if $l = q$. If $X(\xi)$ possesses only one positive eigenvalue $\lambda > 0$, then the flag of isotropic subspaces reduces to a q-dimensional Lagrangian subspace $L = L^{\omega}$ of $(\mathbb{R}^{2q}, \omega)$. The condition $B(X,X) = 1$ moreover implies $\lambda = \frac{1}{\sqrt{2q}}$.

As before one can show that the identification described above is G-equivariant: if $g \in G$ and $\xi \in \partial S$ with corresponding pair $(\lambda(\xi), F(\xi))$ then $\big(\lambda(g \cdot \xi), F(g \cdot \xi)\big) = g \cdot \big(\lambda(\xi), F(\xi)\big)$.

Example 2c. If $S = \mathbb{H}^2 \times \mathbb{H}^2$, for $i = 1, 2$ we denote by $\partial S_i \cong \mathbb{S}^1$ the geometric boundary of the i-th $\mathbb{H}^2$-factor. Then the regular geometric boundary $\partial S^{\mathrm{reg}}$ can be identified with $\partial S_1 \times \partial S_2 \times (0, \pi/2)$. There are two singular boundary strata, one isomorphic to the boundary of the first factor ∂S_1 and one isomorphic to the boundary of the second factor ∂S_2. Hence

$$\partial S \cong \partial S_1 \sqcup \partial S_2 \sqcup \big(\partial S_1 \times \partial S_2 \times (0, \pi/2)\big).$$

3.2 The Furstenberg boundary

Now let us see what happens if we forget about the $\overline{\mathfrak{a}_1^+}$-factor. Recall that M is the centralizer of $\mathfrak{a}$ in K and consider the projection

$$\begin{aligned} \pi^B \colon \partial S^{\mathrm{reg}} &\to K/M, \\ (k, H) &\mapsto kM. \end{aligned}$$

Definition 3.3. We define the *Furstenberg boundary* $\partial^F S$ as $\pi^B(\partial S^{\mathrm{reg}})$.

The Furstenberg boundary has a natural differentiable structure arising from the Lie group structure of K. Geometrically it can be described as the set of equivalence classes of Weyl chambers in S (see [M]), where two Weyl chambers in S are equivalent if and only if they have bounded Hausdorff distance. The following lemma relates the cone topology to the topology of K/M. It is a corollary of Lemma 2.9 in [L1].

Lemma 3.4. *A sequence $(\xi_n) \subset \partial S^{\mathrm{reg}}$ converges to $\xi = (k, H) \in \partial S^{\mathrm{reg}}$ in the cone topology if and only if $\pi^B(\xi_n)$ converges to kM in K/M and the Cartan projections of ξ_n converge to H in $\mathfrak{a}_1^+$.*

Hence π^B is continuous, and $\mathrm{rank}(S) = 1$ if and only if π^B is a homeomorphism.

Moreover, the projection π^B induces an action of G by homeomorphisms on the Furstenberg boundary $K/M = \pi^B(\partial S^{\mathrm{reg}})$. More precisely, if $G = KAN^+$ is the Iwasawa decomposition from Section 2.3 with $A = e^{\mathfrak{a}}$, and π^I the natural projection

$$\begin{aligned} \pi^I \colon G &\to K/M, \\ g = kan &\mapsto kM, \end{aligned}$$

then we have the following

Lemma 3.5. *Let $g \in G$ and $\xi = (k, H) \in \partial S$ with $k \in K$ and $H \in \overline{\mathfrak{a}_1^+}$. If $k' \in K$ is such that $\pi^I(gk) = k'M$, then $g \cdot \xi = (k', H)$.*

In particular, if $\xi \in \partial S^{\mathrm{reg}}$, then $g \cdot \pi^B(\xi) = \pi^B(g \cdot \xi) = k'M$.

Proof. Consider the unit speed geodesic $c := c_{o,\xi}$, i.e. $c(t) = ke^{Ht} \cdot o$ for $t \in \mathbb{R}$. We write $gk = k'an$ with $k' \in K$, $a \in A$ and $n \in N^+$. In order to prove that $g \cdot c(t)$ converges to $(k', H) \in \partial S$ as $t \to \infty$, we let $R \gg 1$ and $\varepsilon > 0$ arbitrary. For $t > R$ we denote by c_t the geodesic emanating from o passing through $g \cdot c(t)$. If

$s_t := d(o, g \cdot c(t))$, then by the triangle inequality $|s_t - t| \le d(o, g \cdot o)$. Using the convexity of the distance function we estimate for $t > R + d(o, g \cdot o)$

$$\begin{aligned} d(k'e^{HR} \cdot o, c_t(R)) &\le \frac{R}{s_t}\big(d(k'e^{Hs_t} \cdot o, g \cdot c(s_t)) + d(g \cdot c(s_t), c_t(s_t))\big) \\ &= \frac{R}{s_t}\big(d(k'e^{Hs_t} \cdot o, gke^{Hs_t} \cdot o) + d(g \cdot c(s_t), g \cdot c(t))\big) \\ &= \frac{R}{s_t}\big(d(k'e^{Hs_t} \cdot o, k'ane^{Hs_t} \cdot o) + \underbrace{d(c(s_t), c(t))}_{=|s_t - t| \le d(o, g \cdot o)}\big) \\ &\le \frac{R}{s_t}\big(d(o, an \cdot o) + d(o, g \cdot o)\big) \end{aligned}$$

since $d(e^{Hs} \cdot o, ane^{Hs} \cdot o) \le d(o, an \cdot o)$ for all $s > 0$. From $s_t \to \infty$ as $t \to \infty$ we get $d(k'e^{HR} \cdot o, c_t(R)) < \varepsilon$ for t sufficiently large. Hence $g \cdot \xi = (k', H)$. $\square$

Example 1. For $S = \mathrm{Pos}_1(n)$, $n \ge 2$, the Furstenberg boundary $\partial^F S$ is identified with the space of regular flags in $\mathbb{R}^n$, i.e. the set of flags $F = (V_1, V_2, \dots, V_n)$ such that $\dim V_i - \dim V_{i-1} = 1$ for all $1 \le i \le n$.

Example 2b. For $S = S_{2q}$, the Furstenberg boundary $\partial^F S$ is identified with the space of complete isotropic flags in $\mathbb{R}^{2q}$, i.e. the set of flags $F = (W_1, W_2, \dots, W_q)$ such that $W_i \subset \mathbb{R}^{2q}$ is an isotropic subspace and $\dim W_i - \dim W_{i-1} = 1$ for all $1 \le i \le q$.

Example 2c. For $S = \mathbb{H}^2 \times \mathbb{H}^2$, the Furstenberg boundary $\partial^F S$ is isomorphic to $\partial S_1 \times \partial S_2$, where ∂S_i denotes the geometric boundary of the i-th $\mathbb{H}^2$-factor, $i \in \{1, 2\}$.

3.3 The Bruhat decomposition

The main reference for this section is [Wa], Chapter 1.2. Given the Iwasawa decomposition $G = KAN^+$ from Section 2.3, we consider the closed subgroup $P = MAN^+ \subset G$. Any subgroup in G conjugate to P is called a *minimal parabolic subgroup*. The homogeneous space G/P can be identified with the Furstenberg boundary K/M via the bijection

$$\begin{aligned} \bar{\kappa}\colon G/P &\to K/M, \\ gP &\mapsto \pi^I(g). \end{aligned}$$

The Bruhat decomposition gives a cell decomposition of G/P, hence induces a cell decomposition of the Furstenberg boundary which we will describe geometrically in the next section.

Recall that the factor group $W = M^*/M$ is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. We denote by $w_* \in W$ the unique element such that $\mathrm{Ad}(m_{w_*})(-\mathfrak{a}^+) = \mathfrak{a}^+$ for any representative m_{w_*} of w_* in M^*, and put

$$\mathfrak{n}^- := \mathrm{Ad}(m_{w_*})\mathfrak{n}^+ = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}.$$

For $w \in W$ represented by $m_w \in M^*$ we set

$$\mathfrak{u}_w := \mathfrak{n}^+ \cap \mathrm{Ad}(m_w)\mathfrak{n}^- \subset \mathfrak{n}^+ \quad \text{and} \quad U_w := e^{\mathfrak{u}_w}.$$

The Bruhat decomposition of G with respect to the minimal parabolic subgroup P is the disjoint union

$$G = \bigsqcup_{w\in W} N^+ m_w P = \bigsqcup_{w\in W} U_w m_w P. \tag{3.3}$$

Notice that the orbit corresponding to $w_* \in W$ is parametrized by $N^+ = U_{w_*}$, and the restriction of the above bijection $\bar\kappa$ to $N^+ m_{w_*} P$ defines a map

$$\begin{aligned} \kappa\colon N^+ &\to K/M, \\ n &\mapsto \bar\kappa(n m_{w_*} P). \end{aligned}$$

Geometrically, this map can be interpreted in the following way: if $n \in N^+$, then $\kappa(n) \in K/M$ is the unique element such that the Weyl chamber $\kappa(n)e^{\mathfrak{a}^+}\cdot o$ is equivalent to the Weyl chamber $ne^{-\mathfrak{a}^+}\cdot o$. The following property of the map κ is well-known:

Proposition 3.6 ([H], Corollary IX.1.9). *The map κ is a diffeomorphism onto an open submanifold of K/M whose complement consists of finitely many disjoint manifolds of strictly lower dimension.*

It follows that the orbit $N^+ m_{w_*} P$ is a dense and open submanifold of G/P. We will call a G-translate of the set $\kappa(N^+) = \bar\kappa(N^+ m_{w_*} P) \subset K/M$ a *big cell* of the Furstenberg boundary.

Example 1. For simplicity we treat the case $n = 3$. Recall the Iwasawa decomposition $G = KAN^+$ of $G = \mathrm{SL}(3,\mathbb{R})$ from (2.3), where $K = \mathrm{SO}(3)$, A denotes the set of diagonal matrices with positive entries in $\mathrm{SL}(3,\mathbb{R})$, and N^+ the set of upper diagonal $(n\times n)$-matrices with 1's in the diagonal. Here the centralizer of $\mathfrak{a}$ in K is the finite set

$$M = \{\mathrm{Diag}(1,1,1), \mathrm{Diag}(-1,-1,1), \mathrm{Diag}(-1,1,-1), \mathrm{Diag}(1,-1,-1)\},$$

and $P = MAN^+$ is a minimal parabolic subgroup. The Weyl group W is represented by the set of matrices

$$\Big\{e = I_3,\ w_1 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\ w_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},$$
$$w_3 = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\ w_4 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},\ w_* = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}\Big\}$$

One easily computes the sets U_w as follows: $U_e = \{e\}$, $U_{w_*} = N^+$,

$$U_{w_1} = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad U_{w_3} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x, z \in \mathbb{R} \right\},$$
$$U_{w_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}, \quad U_{w_4} = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{R} \right\}.$$

Hence for $\mathrm{SL}(3, \mathbb{R})$ the Bruhat decomposition yields 6 cells, one isomorphic to a point, two isomorphic to $\mathbb{R}$, two isomorphic to $\mathbb{R}^2$, and the maximal one isomorphic to $\mathbb{R}^3$.

Example 2b. We consider $q = 2$, and use the Iwasawa decomposition $G = KAN^+$ described in Section 2.3. Here the centralizer of $\mathfrak{a} = \{\mathrm{Diag}(t_1, t_2, -t_1, -t_2) : t_1, t_2 \in \mathbb{R}\}$ in K is the finite set

$$M = \{I_4, \mathrm{Diag}(1, -1, 1, -1), \mathrm{Diag}(-1, 1, -1, 1), -I_4\},$$

and $P = MAN^+$ is a minimal parabolic subgroup. The following table gives a set of representatives w of the Weyl group W and describes the corresponding sets $\mathfrak{u}_w \subseteq \mathfrak{n}^+$:

w	$\mathfrak{u}_w$
$e = \mathrm{Diag}(1, 1, 1, 1)$	0
$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\left\{ \left(\begin{array}{cc\|cc} 0 & b & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 0 \\ \multicolumn{2}{c\|}{0} & -b & 0 \end{array} \right) : b \in \mathbb{R} \right\}$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$\left\{ \left(\begin{array}{c\|c} 0 & \begin{matrix} 0 & 0 \\ 0 & z \end{matrix} \\ \hline 0 & 0 \end{array} \right) : z \in \mathbb{R} \right\}$
$\begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\left\{ \left(\begin{array}{cc\|cc} 0 & b & x & 0 \\ 0 & 0 & 0 & 0 \\ \hline & & 0 & 0 \\ \multicolumn{2}{c\|}{0} & -b & 0 \end{array} \right) : b, x \in \mathbb{R} \right\}$
$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\left\{ \left(\begin{array}{c\|c} 0 & \begin{matrix} 0 & y \\ y & z \end{matrix} \\ \hline 0 & 0 \end{array} \right) : y, z \in \mathbb{R} \right\}$
$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\left\{ \left(\begin{array}{c\|c} 0 & \begin{matrix} x & y \\ y & z \end{matrix} \\ \hline 0 & 0 \end{array} \right) : x, y, z \in \mathbb{R} \right\}$
$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\left\{ \left(\begin{array}{cc\|cc} 0 & b & x & y \\ 0 & 0 & y & 0 \\ \hline & & 0 & 0 \\ \multicolumn{2}{c\|}{0} & -b & 0 \end{array} \right) : b, x, y \in \mathbb{R} \right\}$
$w_* = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$	$\mathfrak{n}^+$

Hence for $\mathrm{Sp}(4, \mathbb{R})$ the Bruhat decomposition yields 8 cells, one isomorphic to a point, two isomorphic to $\mathbb{R}$, two isomorphic to $\mathbb{R}^2$, two isomorphic to $\mathbb{R}^3$ and the maximal one isomorphic to $\mathbb{R}^4$.

Example 2c. We have seen that for $S = \mathbb{H}^2 \times \mathbb{H}^2$ the Furstenberg boundary is given by $\partial^F S = \partial S_1 \times \partial S_2$. Hence $\partial^F S \cong (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\})$. Here the Bruhat decomposition is isomorphic to the decomposition

$$\partial^F S = \{(\infty, \infty)\} \sqcup (\{\infty\} \times \mathbb{R}) \sqcup (\mathbb{R} \times \{\infty\}) \sqcup (\mathbb{R} \times \mathbb{R}).$$

3.4 Visibility at infinity

If the rank of S equals one, or, more generally, if S is a Hadamard manifold with a negative upper bound on the sectional curvature, then any pair of distinct points in the geometric boundary can be joined by a geodesic. In a symmetric space of higher rank this fails to be true. In this section we will describe the set of points in the geometric boundary ∂S of a globally symmetric space S which can be joined to a given point $\xi \in \partial S$ by a geodesic.

Definition 3.7. The *visibility set at infinity* viewed from $\xi \in \partial S$ is the set

$$\mathrm{Vis}^\infty(\xi) := \{\eta \in \partial S \mid \text{there exists a geodesic } c \text{ such that } c(-\infty) = \xi,\ c(\infty) = \eta\}.$$

It is clear that for $g \in \mathrm{Is}(S)$ we have $\mathrm{Vis}^\infty(g \cdot \xi) = g \cdot \mathrm{Vis}^\infty(\xi)$. Moreover, we have the following description of $\mathrm{Vis}^\infty(\xi)$, which is Proposition 2.21.13 (2) in [E]. We include a proof here for the convenience of the reader.

Proposition 3.8. *If* $\mathrm{Stab}_G(\xi) \subset G$ *denotes the stabilizer of a point* $\xi \in \partial S$*, then*

$$\mathrm{Vis}(\xi) = \mathrm{Stab}_G(\xi) \cdot c_{o,\xi}(-\infty).$$

Proof. By transitivity of the G-action and the fact that $g \cdot \mathrm{Vis}^\infty(\xi) = \mathrm{Vis}^\infty(g \cdot \xi)$ for $g \in G$ and $\xi \in \partial S$ we can assume that $\xi \in \partial S$ is stabilized by the minimal parabolic subgroup $P = MAN^+$, so $P \subseteq \mathrm{Stab}_G(\xi)$. If $\eta \in \mathrm{Vis}^\infty(\xi)$, then there exists a geodesic $c \subset S$ such that $c(-\infty) = \xi$ and $c(\infty) = \eta$. Let $g \in G$ be such that $c(0) = g \cdot o$. Using the Iwasawa decomposition we may write $g = nak$ with $n \in N^+, a \in e^{\mathfrak{a}}$ and $k \in K$. Then $c(0) = nak \cdot o = na \cdot o$. The geodesic c_0 defined by $c_0(t) := (na)^{-1} \cdot c(t) = a^{-1}n^{-1} \cdot c(t)$ satisfies $c_0(-\infty) = c(-\infty) = \xi$ because N^+ and A stabilize ξ. So $c_0(0) = (na)^{-1} \cdot c(0) = o$ implies $c_0(t) = c_{o,\xi}(-t)$ for all $t \in \mathbb{R}$. We conclude

$$\eta = c(\infty) = na \cdot c_0(\infty) = na \cdot c_{o,\xi}(-\infty),$$

hence $\eta \in \mathrm{Stab}_G(\xi) \cdot c_{o,\xi}(-\infty)$.

Conversely let $p \in \mathrm{Stab}_G(\xi)$ and set $\eta := p \cdot c_{o,\xi}(-\infty)$. If c is the geodesic defined by $c(t) := p \cdot c_{o,\xi}(-t)$ for $t \in \mathbb{R}$, then we have $c(\infty) = \eta$ and $c(-\infty) = p \cdot c_{o,\xi}(\infty) = p \cdot \xi = \xi$ because p fixes ξ. Hence $\eta \in \mathrm{Vis}^\infty(\xi)$. $\square$

The following lemma relates the visibility set of regular points to our coordinates introduced in Section 3.1 and the map κ from Section 3.3. Even though it is a direct consequence of [L1], Corollary 2.15, we include the proof here for the convenience of the reader. Recall the definition of the opposition involution (2.4).

Lemma 3.9. *If $\xi \in \partial S^{\mathrm{reg}}$ is stabilized by the minimal parabolic subgroup $P \subset G$ and possesses the Cartan projection $H \in \mathfrak{a}_1^+$, then*

$$\mathrm{Vis}(\xi) = \{(k, \iota(H)) : kM \in \kappa(N^+)\}.$$

Proof. Let $k \in K$ be such that $kM = \kappa(n)$ with $n \in N^+$. Consider the geodesic $c(t) := n \cdot c_{o,\xi}(-t)$ which satisfies $c(-\infty) = \xi$ because N^+ stabilizes ξ. If $m_{w_*} \in M^*$ is a representative of $w_* \in W$, we have

$$c(t) = ne^{-Ht} \cdot o = nm_{w_*}e^{\iota(H)t} \cdot o,$$

hence by the property of the map κ the geodesic rays $c(t)$, $t > 0$, and $ke^{\iota(H)t} \cdot o$, $t > 0$, are equivalent. This shows $c(\infty) = (k, \iota(H))$, so we conclude $(k, \iota(H)) \in \mathrm{Vis}^\infty(\xi)$.

Conversely, let $\eta \in \mathrm{Vis}^\infty(\xi)$ and choose a geodesic c in S with $c(\infty) = \eta$ and $c(-\infty) = \xi$. From the proof of Proposition 3.8 we know that there exist $n \in N^+$ and $a \in e^{\mathfrak{a}}$ such that the geodesic c_0 defined for $t \in \mathbb{R}$ by $c_0(t) := (na)^{-1} \cdot c(t) = a^{-1}n^{-1} \cdot c(t)$ satisfies $c_0(t) = c_{o,\xi}(-t) = e^{-Ht} \cdot o$. Moreover, since A also stabilizes $c_{o,\xi}(-\infty)$ we conclude

$$\eta = c(\infty) = na \cdot c_0(\infty) = na \cdot c_{o,\xi}(-\infty) = n \cdot c_{o,\xi}(-\infty) = n \cdot c_0(\infty).$$

If $k \in K$ is an angular projection of η, then $\eta = (k, \iota(H))$, and the geodesic ray $ke^{\iota(H)t} \cdot o$, $t > 0$, is equivalent to the geodesic ray $ne^{-Ht} \cdot o$, $t > 0$. Hence by definition of the map κ we have $kM = \kappa(n)$. $\square$

Since the opposition involution preserves the set $\mathfrak{a}^+$ of regular elements in $\overline{\mathfrak{a}^+}$, this lemma in particular implies that the visibility set at infinity viewed from a regular boundary point is contained in the regular boundary. This allows the following

Definition 3.10. The *Bruhat visibility set* viewed from $\xi \in \partial S^{\mathrm{reg}}$ is the image of $\mathrm{Vis}^\infty(\xi)$ under the projection $\pi^B : \partial S^{\mathrm{reg}} \to K/M$, i.e.

$$\mathrm{Vis}^B(\xi) = \pi^B(\mathrm{Vis}^\infty(\xi)).$$

We remark that if $\mathrm{rank}(S) = 1$, then $\mathrm{Vis}^B(\xi) \cong \mathrm{Vis}^\infty(\xi) = \partial S \setminus \{\xi\}$ for all $\xi \in \partial S$. In general, an immediate consequence of Lemma 3.9 is the fact that $\mathrm{Vis}^B(\xi)$ can be identified with the nilpotent Lie group N^+ or an arbitrary orbit $N^+ \cdot x$, $x \in S$. Moreover, all Bruhat visibility sets are open and dense submanifolds of K/M by Proposition 3.6.

Example 1. Let $S = \mathrm{Pos}_1(n)$, $n \geq 3$. Assume first that ξ is stabilized by $P = MAN^+ \subset G$. Then there exist $l \in \{2, 3, \ldots, n\}$, $(m_1, m_2, \ldots, m_l) \in \mathbb{N}^l$ with $\sum_{i=1}^l m_i = n$ such that $\lambda(\xi) = (\lambda_1, \lambda_2, \ldots \lambda_l) \in \mathbb{R}^l$ and $F(\xi) = (U_1, U_1, \ldots, U_l)$ is the standard flag in $\mathbb{R}^n$ determined by $(m_1, m_2, \ldots, m_l)$ via (3.1).

For $1 \leq i \leq l$ we denote by $U_i^\perp$ the orthogonal complement of U_i and remark that $\zeta := c_{o,\xi}(-\infty)$ corresponds to the eigenvalue-flag pair $\big(\lambda(\zeta), F(\zeta)\big)$ with $\lambda(\zeta) = (-\lambda_l, -\lambda_{l-1}, \dots, -\lambda_1)$ and $F(\zeta) = (U_l^\perp, U_{l-1}^\perp, \dots, U_1^\perp)$.

We will say that two flags $F = (V_1, V_2, \dots, V_l)$, $F' = (W_1, W_2, \dots, W_k)$ are *in opposition* if $k = l$ and $\mathbb{R}^n = V_i \oplus W_{l-i+1}$, $1 \leq i \leq l-1$. For ξ and ζ as above clearly $F(\xi)$ and $F(\zeta)$ are in opposition. Moreover, if $g \in G = \mathrm{SL}(n, \mathbb{R})$, then $g \cdot F$ and $g \cdot F'$ are in opposition if and only if F and F' are.

By Proposition 3.8 we have $\mathrm{Vis}^\infty(\xi) = \mathrm{Stab}_G(\xi) \cdot \zeta$, and $g \in G$ stabilizes ξ if and only if g leaves invariant each of the eigenspaces for the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_l$ of $X(\xi) \in \mathfrak{p}$. This is equivalent to $g \cdot U_i = U_i$ for all $1 \leq i \leq l$. So we conclude that $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if there exists $g \in \mathrm{Stab}_G(\xi)$ such that

$$\lambda(\eta) = \lambda(g \cdot \zeta) = \lambda(\zeta) = (-\lambda_l, -\lambda_{l-1}, \dots, -\lambda_1),$$

and $F(\eta) = F(g \cdot \zeta) = (g \cdot U_l^\perp, g \cdot U_{l-1}^\perp, \dots, g \cdot U_1^\perp)$. This second condition is satisfied if and only if $F(\eta)$ and $F(\xi)$ are in opposition.

If $\xi \in \partial S$ is arbitrary, there exists $g \in G$ such that $g \cdot \xi$ is stabilized by P. So there exist $l \in \{2, 3, \dots, n\}$, $(m_1, m_2, \dots, m_l) \in \mathbb{N}^l$ with $\sum_{i=1}^l m_i = n$ such that $\lambda(g \cdot \xi) = (\lambda_1, \lambda_2, \dots \lambda_l) \in \mathbb{R}^l$ and $F(g \cdot \xi) = (U_1, U_1, \dots, U_l)$ is the standard flag in $\mathbb{R}^n$ determined by $(m_1, m_2, \dots, m_l)$ via (3.1). Since $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $g \cdot \eta \in \mathrm{Vis}^\infty(g \cdot \xi)$, this shows that $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $\lambda(\eta) = (-\lambda_l, -\lambda_{l-1}, \dots, -\lambda_1)$, and $F(\eta)$ is in opposition to $F(\xi)$. Summarizing we have $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if the following two conditions are satisfied:

(a) If $\lambda(\xi) = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\lambda(\eta) = (\mu_1, \mu_2, \dots, \mu_k)$, then $k = l$ and $\mu_i = -\lambda_{l-i+1}$ for all $i \in \{1, 2, \dots l\}$.

(b) $F(\xi)$ and $F(\eta)$ are in opposition.

This immediately implies that $\pi^B(\eta) \in \mathrm{Vis}^B(\xi)$, $\xi \in \partial S^{\mathrm{reg}}$, if and only if the regular flags $F(\xi)$ and $F(\eta)$ are in opposition.

Example 2b. Consider $S = S_{2q}$ for $q \geq 2$. As before we first assume that ξ is stabilized by $P = MAN^+ \subset G = \mathrm{Sp}(2q, \mathbb{R})$. Then there exist $1 \leq l \leq q$, $(m_1, m_2, \dots, m_l) \in \mathbb{N}^l$ with $\sum_{i=1}^l m_i \leq q$ such that $\lambda(\xi) = (\lambda_1, \lambda_2, \dots \lambda_l) \in \mathbb{R}^l$ and $F(\xi) = (U_1, U_2, \dots, U_l)$ is the isotropic standard flag in $(\mathbb{R}^{2q}, \omega)$ determined by $(m_1, m_2, \dots, m_l)$ via (3.2).

Let $m_* \in G$ be the element defined by the matrix $\begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}$. Here $\zeta := c_{o,\xi}(-\infty)$ corresponds to the pair $\big(\lambda(\zeta), F(\zeta)\big)$ with $\lambda(\zeta) = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $F(\zeta) = m_* \cdot F(\xi)$. Moreover, each of the linear subspaces $U_i \oplus m_* \cdot U_i$, $1 \leq i \leq l$, is a symplectic subspace of $(\mathbb{R}^{2q}, \omega)$, i.e. ω restricted to $U_i \oplus m_* \cdot U_i$ is non-degenerate.

Motivated by this property we will say that two flags $F = (V_1, V_2, \dots, V_l)$, $F' = (W_1, W_2, \dots, W_k)$ of isotropic subspaces are *complementary* if $k = l$ and $V_i \oplus W_i$, $1 \leq i \leq l$, is a symplectic subspace of $(\mathbb{R}^{2q}, \omega)$. Notice that this necessarily implies $\dim W_i = \dim V_i$ for all $i \in \{1, 2, \dots, l\}$. Clearly the isotropic flags $F(\xi)$ and

$F(\zeta)$ from above are complementary; moreover, if $g \in G$, then $g \cdot F$ and $g \cdot F'$ are complementary if and only if F and F' are.

Now Proposition 3.8 implies $\mathrm{Vis}^\infty(\xi) = \mathrm{Stab}_G(\xi) \cdot \zeta$, and $g \in G$ stabilizes ξ if and only if g leaves invariant each of the isotropic eigenspaces for the positive eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_l$ of $X(\xi)$. Hence we conclude that $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if there exists $g \in \mathrm{Stab}_G(\xi)$ such that $\lambda(\eta) = \lambda(g \cdot \zeta) = \lambda(\zeta) = \lambda(\xi)$, and $F(\eta) = F(g \cdot \zeta) = g \cdot F(\zeta)$. The latter condition is satisfied if and only if $F(\eta)$ is complementary to $F(\xi)$.

If $\xi \in \partial S$ is arbitrary, there exists $g \in G$ such that $g \cdot \xi$ is stabilized by P. So there exist $l \in \{1, 2, \dots, q\}$, $(m_1, m_2, \dots, m_l) \in \mathbb{N}^l$ with $\sum_{i=1}^l m_i \leq q$ such that $\lambda(g \cdot \xi) = (\lambda_1, \lambda_2, \dots \lambda_l) \in \mathbb{R}^l$ and $F(g \cdot \xi) = (U_1, U_1, \dots, U_l)$ is the isotropic standard flag determined by $(m_1, m_2, \dots, m_l)$ via (3.2). Since $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $g \cdot \eta \in \mathrm{Vis}^\infty(g \cdot \xi)$ this shows that $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $\lambda(g \cdot \eta) = \lambda(\zeta) = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $F(\xi)$, $F(\eta)$ are complementary. We conclude that $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if the following conditions are satisfied:

(a) If $\lambda(\xi) = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $\lambda(\eta) = (\mu_1, \mu_2, \dots, \mu_k)$, then $k = l$ and $\mu_i = \lambda_i$ for all $i \in \{1, 2, \dots l\}$.

(b) The isotropic flags associated to ξ and η are complementary.

This implies in particular that $\pi^B(\eta) \in \mathrm{Vis}^B(\xi)$, $\xi \in \partial S^{\mathrm{reg}}$, if and only if the complete isotropic flags of ξ and η are complementary.

Example 2c. If $S = \mathbb{H}^2 \times \mathbb{H}^2$ we have seen that

$$\partial S = \partial S_1 \sqcup \partial S_2 \sqcup \partial S^{\mathrm{reg}},$$

where $\partial S^{\mathrm{reg}} = \partial S_1 \times \partial S_2 \times (0, \pi/2)$, and ∂S_1, ∂S_2 are the two singular boundary strata. For $i \in \{1, 2\}$, $\xi \in \partial S_i$, we have $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $\eta \in \partial S_i$ and $\eta \neq \xi$.

If $\xi = (\xi_1, \xi_2, \theta) \in \partial S^{\mathrm{reg}}$, then $\eta \in \mathrm{Vis}^\infty(\xi)$ if and only if $\eta = (\eta_1, \eta_2, \varphi) \in \partial S^{\mathrm{reg}}$ with $\eta_1 \neq \xi_1$, $\eta_2 \neq \xi_2$ and $\varphi = \theta$.

3.5 Busemann functions and distances

In this final section we discuss Busemann functions and how they can be used to construct a family of G-invariant Finsler pseudo-distances on S for which the flats are isomorphic to a pseudo-normed vector space. For more details about G-invariant Finsler structures on symmetric spaces we refer the reader to P. Planche's thesis ([P]).

Let $x, y \in S$, $\xi \in \partial S$, and $c \subset S$ a geodesic ray in the class of ξ. We put

$$\mathcal{B}_\xi(x, y) := \lim_{s \to \infty} \big(d(x, c(s)) - d(y, c(s))\big).$$

This number is independent of the chosen ray c, and the function

$$\begin{aligned} \mathcal{B}_\xi(\cdot, y) \colon S &\to \mathbb{R}, \\ x &\mapsto \mathcal{B}_\xi(x, y), \end{aligned}$$

is called the *Busemann function* centered at ξ based at y (see also Chapter II of [Ba]). It satisfies the following properties:

Proposition 3.11. *For all* $\xi \in \partial S$, $x, y, z \in S$, $g \in G := \mathrm{Is}^o(S)$ *we have*

(1) $\mathcal{B}_{g\cdot\xi}(g\cdot x, g\cdot y) = \mathcal{B}_\xi(x,y)$,

(2) $\mathcal{B}_\xi(x,z) = \mathcal{B}_\xi(x,y) + \mathcal{B}_\xi(y,z)$,

(3) $|\mathcal{B}_\xi(x,y)| \le d(x,y)$,

(4) $\mathcal{B}_\xi(x,y) = d(x,y)$ *if and only if* $\xi = c_{x,y}(\infty)$.

Using Busemann functions we introduce an important family of (possibly non-symmetric) pseudo-distances.

Definition 3.12. Let $\xi \in \partial S$. We define the *directional distance* of the ordered pair $(x,y) \in S \times S$ with respect to the subset $G\cdot\xi \subseteq \partial S$ by

$$\begin{aligned}\mathcal{B}_{G\cdot\xi}\colon S\times S &\to \mathbb{R},\\ (x,y) &\mapsto \mathcal{B}_{G\cdot\xi}(x,y) := \sup_{g\in G} \mathcal{B}_{g\cdot\xi}(x,y).\end{aligned}$$

Notice that in rank one symmetric spaces $G\cdot\xi = \partial S$ and for $x, y \in S$ we have

$$d(x,y) = \mathcal{B}_{G\cdot\xi}(x,y) = \sup_{\eta\in\partial S} \mathcal{B}_\eta(x,y) \ge \mathcal{B}_{c_{x,y}(\infty)}(x,y) = d(x,y),$$

hence $\mathcal{B}_{G\cdot\xi}$ equals the Riemannian distance d for any $\xi \in \partial S$. In general, the corresponding estimate for the Busemann functions implies

$$\mathcal{B}_{G\cdot\xi}(x,y) \le d(x,y) \quad \text{for all } \xi\in\partial S \text{ and all } x,\ y \in S.$$

Moreover, $\mathcal{B}_{G\cdot\xi}$ is a (possibly non-symmetric) G-invariant pseudo-distance on S (for a proof see [L1], Proposition 3.7), and we have

$$\mathcal{B}_{G\cdot\xi}(x,y) = d(x,y)\cdot \sup_{g\in G} \cos\angle_x(y, g\xi).$$

In particular, if $G = Ke^{\overline{\mathfrak{a}^+}}K$ is a Cartan decomposition, $H_\xi \in \overline{\mathfrak{a}_1^+}$ the Cartan projection of ξ, and $H(x,y) \in \overline{\mathfrak{a}^+}$ the Cartan vector of the ordered pair (x,y) according to Definition 2.28, then

$$\mathcal{B}_{G\cdot\xi}(x,y) = \langle\langle H_\xi, H(x,y)\rangle\rangle = B\big(H_\xi, H(x,y)\big) \quad \text{for all } x, y \in S. \tag{3.4}$$

This shows in particular that the flats of S are isomorphic to $\mathbb{R}^r$ endowed with a pseudo-norm.

Moreover, from the remark following Definition 2.28 we know that

$$\mathcal{B}_{G\cdot\xi}(y,x) = \langle\langle H_\xi, \iota(H(x,y))\rangle\rangle = \langle\langle \iota(H_\xi), H(x,y)\rangle\rangle,$$

because ι is an involution and, by $\mathrm{Ad}(K)$-invariance of the Killing form, preserves the scalar product. So $\mathcal{B}_{G\cdot\xi}$ is symmetric if and only if the Cartan projection H_ξ of

ξ satisfies $\iota(H_\xi) = H_\xi$. This clearly always holds when ι is the identity; so all the directional distances are symmetric e.g. in S_{2q}, $q \geq 1$, and $\mathbb{H}^2 \times \mathbb{H}^2$.

Example 2c. If $S = \mathbb{H}^2 \times \mathbb{H}^2$ we have seen that

$$\partial S = \partial S_1 \sqcup \partial S_2 \sqcup \partial S^{\mathrm{reg}},$$

where $\partial S^{\mathrm{reg}} = \partial S_1 \times \partial S_2 \times (0, \pi/2)$, and ∂S_1, ∂S_2 are the two singular boundary strata.

If $\xi = (\xi_1, \xi_2, \theta) \in \partial S^{\mathrm{reg}}$ one can easily deduce from the definition of the Busemann functions that for $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$\mathcal{B}_\xi(x, y) = \cos\theta \cdot \mathcal{B}_{\xi_1}(x_1, y_1) + \sin\theta \cdot \mathcal{B}_{\xi_2}(x_2, y_2).$$

For the directional distance we therefore get by definition and the remark about rank one symmetric spaces

$$\mathcal{B}_{G\cdot\xi}(x, y) = \cos\theta \cdot d_1(x_1, y_1) + \sin\theta \cdot d_2(x_2, y_2),$$

where for $i \in \{1, 2\}$, d_i denotes the Riemannian distance in the i-th $\mathbb{H}^2$-factor. Hence for $\xi \in \partial S^{\mathrm{reg}}$ the directional distance is a proper distance function.

If $\xi \in \partial S_i$, $i \in \{1, 2\}$, we similarly obtain

$$\mathcal{B}_\xi(x, y) = \mathcal{B}_\xi(x_i, y_i) \quad \text{and} \quad \mathcal{B}_{G\cdot\xi}(x, y) = d_i(x_i, y_i).$$

In particular, $\mathcal{B}_\xi$ is symmetric, but only a pseudo-distance.

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